Review guide for midterm 1.

February 2, 2009

1 Basics.

First we cover the basic definitions and then we go over related problems. Note that the material for the actual midterm may include material from the review guide for midterm 2. Before the exam, view the updated web course web page for the exact material covered on midterm 1.

Definition 1 Let n be a positive integer. Then the cartesian product of n copies of the real number line $\mathbb{R}$ is:

$$\mathbb{R}^n = \mathbb{R} \times \mathbb{R} \times \ldots \times \mathbb{R} = \{(a_1, a_2, \ldots, a_n \mid a_j \in \mathbb{R})\},$$

which is the set of all ordered n-tuples of real numbers.

Example 2 (a) $\mathbb{R}^2 = \mathbb{R} \times \mathbb{R} = \{(a_1, a_2) \mid a_i \in \mathbb{R}\}$ is the Euclidean plane.

(b) $\mathbb{R}^3 = \mathbb{R} \times \mathbb{R} \times \mathbb{R} = \{(a_1, a_2, a_3) \mid a_i \in \mathbb{R}^3\}$ is Euclidian three-space.

There are two standard notations for points in $\mathbb{R}^3$, or more generally $\mathbb{R}^n$. If $P \in \mathbb{R}^3$, then $P = (a_1, a_2, a_3)$ for some scalars a_1, a_2, a_3. The book also denotes this point by writing $P(a_1, a_2, a_3)$. The scalar a_1 is called the <u>x-coordinate</u> of P, a_2 is called the <u>y-coordinate</u> of P and a_3 is called the <u>z-coordinate</u> of P.

Example 3 The point $P = (1, 0, 7)$ in $\mathbb{R}^3$ can also be written as $P(1, 0, 7)$. Its z-coordinate is 7.

Definition 4 (a) Given points $P = (x_1, y_1, z_1)$ and $Q = (x_2, y_2, z_2)$ in $\mathbb{R}^3$, then $\overrightarrow{PQ} = \langle x_2 - x_1, y_2 - y_1, z_2 - z_1 \rangle$ denotes the *arrow* or *vector* based at P with terminal point Q.

(b) If $\lambda \in \mathbb{R}$ is a scalar and $\mathbf{v} = \langle a, b, c \rangle$ is a vector, then consider the new vector $\lambda\mathbf{v} = \langle \lambda a, \lambda b, \lambda c \rangle$; if $\lambda > 0$, then $\lambda\mathbf{v}$ is the vector pointed in the direction $\mathbf{v}$ and has length $\lambda|\mathbf{v}|$; if $\lambda < 0$, then $\lambda\mathbf{v}$ is the vector pointed in the opposite direction of $\mathbf{v}$ with length $|\lambda||\mathbf{v}|$.

(c) If $\mathbf{u} = \langle x_1, y_1, z_1 \rangle$ and $\mathbf{v} = \langle x_2, y_2, z_2 \rangle$, then $\mathbf{u} + \mathbf{v} = \langle x_1 + x_2, y_1 + y_2, z_1 + z_2 \rangle$. In other words, vectors add by adding their coordinates.

Definition 5 If $\mathbf{a} = \langle x_1, y_1, z_1 \rangle$ and $\mathbf{b} = \langle x_2, y_2, z_2 \rangle$, then the **dot product** of $\mathbf{a}$ and $\mathbf{b}$ is:

$$\mathbf{a} \cdot \mathbf{b} = x_1 x_2 + y_1 y_2 + z_1 z_2.$$

Example 6 The dot product of $\langle 1, 2, 3 \rangle$ and $\langle 1, 0, 7 \rangle$ is

$$\langle 1, 2, 3 \rangle \cdot \langle 1, 0, 7 \rangle = 1 + 0 + 21 = 22.$$

It turns out that the length of a vector can be found by using the dot product and it satisfies some nice algebraic properties listed in the next two theorems.

Theorem 7 Let $\mathbf{a} = \langle x_1, y_1, z_1 \rangle$ be vector and let $P = (x_2, y_2, z_2)$, $Q = (x_3, y_3, z_3)$ be points. Then:

1. The length of $\mathbf{a}$ is $|\mathbf{a}| = \sqrt{\mathbf{a} \cdot \mathbf{a}} = \sqrt{x_1^2 + y_1^2 + z_1^2}$.

2. The distance $d(P, Q)$ from the point P to the point Q is:

$$d(P, Q) = |\overrightarrow{PQ}| = \sqrt{(x_3 - x_2)^2 + (y_3 - y_2)^2 + (z_3 - z_2)^2}$$

Theorem 8 (Basic algebraic properties of dot product) Let $\mathbf{a} = \langle x_1, y_1, z_1 \rangle$, $\mathbf{b} = \langle x_2, y_2, z_2 \rangle$, $\mathbf{c} = \langle x_3, y_3, z_3 \rangle$ be vectors and let λ be a scalar.

1. $\mathbf{a} \cdot \mathbf{b} = \mathbf{b} \cdot \mathbf{a}$.

2. $(\mathbf{a} + \mathbf{b}) \cdot \mathbf{c} = \mathbf{a} \cdot \mathbf{c} + \mathbf{b} \cdot \mathbf{c}$.

3. $(\lambda \mathbf{a}) \cdot \mathbf{b} = \lambda(\mathbf{a} \cdot \mathbf{b})$.

Definition 9 The sphere in $\mathbb{R}^3$ with center $C = (x_0, y_0, z_0)$ and radius r is the set where $(x - x_0)^2 + (y - y_0)^2 + (z - z_0)^2 = r^2$. Note that this sphere is geometrically the set of points (x, y, z) of distance r from the point (x_0, y_0, z_0).

Example 10 Consider the subset of $\mathbb{R}^3$ defined by $x^2 + y^2 + 6y + z^2 + 2z = 26$. By completing the square, we have

$$x^2 + (y^2 + 6y + 9) + (z^2 + 2z + 1) = 26 + 10 = 36,$$

which simplifies to be

$$x^2 + (y + 3)^2 + (z + 1)^2 = 6^2.$$

So this set is the sphere centered at $(0, -3, -1)$ of radius 6.

For convenience, it is useful to pick out the special unit vectors pointed respectively along the positive x, y and z-axes, as given in the next definition.

Definition 11 We define the standard basic vectors for $\mathbb{R}^3$ as follows: $\mathbf{i} = \langle 1, 0, 0 \rangle$, $\mathbf{j} = \langle 0, 1, 0 \rangle$, $\mathbf{k} = \langle 0, 0, 1 \rangle$. Note that the vector $\langle a, b, c \rangle$ can be expressed by $\langle a, b, c \rangle = a\mathbf{i} + b\mathbf{j} + c\mathbf{k}$.

For nonzero vectors $\mathbf{a}, \mathbf{b}$

$$\mathbf{a} \cdot \mathbf{b} = |\mathbf{a}||\mathbf{b}| \cos(\theta),$$

where $\theta \in [0, \pi]$ is the angle between the vectors. It follows that:

1. $\mathbf{a}$ and $\mathbf{b}$ are orthogonal or perpendicular if and only if $\mathbf{a} \cdot \mathbf{b} = 0$.

2. The angle θ between $\mathbf{a}$ and $\mathbf{b}$ is an acute angle if and only if $\mathbf{a} \cdot \mathbf{b} > 0$

3. The angle θ between $\mathbf{a}$ and $\mathbf{b}$ is an obtuse angle if and only if $\mathbf{a} \cdot \mathbf{b} < 0$.

4. $\cos(\theta) = \frac{\mathbf{a} \cdot \mathbf{b}}{|\mathbf{a}||\mathbf{b}|}$.

5. $\theta = \arccos(\frac{\mathbf{a} \cdot \mathbf{b}}{|\mathbf{a}||\mathbf{b}|}) = \cos^{-1}(\frac{\mathbf{a} \cdot \mathbf{b}}{|\mathbf{a}||\mathbf{b}|})$ In particular, if $\mathbf{a}$ and $\mathbf{b}$ are unit vectors, then $\theta = \arccos(\mathbf{a} \cdot \mathbf{b})$.

Definition 12 1. The scalar projection (component) of $\mathbf{b}$ onto $\mathbf{a}$ is $\text{comp}_{\mathbf{a}}\mathbf{b} = \frac{\mathbf{a}\cdot\mathbf{b}}{|\mathbf{a}|}$. In particular, if $\mathbf{a}$ is a unit vector, then $\text{comp}_{\mathbf{a}}\mathbf{b} = \mathbf{a}\cdot\mathbf{b}$.

2. The vector projection of $\mathbf{b}$ onto (in the direction of) $\mathbf{a}$ is a $\text{proj}_{\mathbf{a}}\mathbf{b} = (\frac{\mathbf{a}\cdot\mathbf{b}}{\mathbf{a}\cdot\mathbf{a}})\mathbf{a}$. In particular, if $\mathbf{a}$ is a unit vector, then $\text{proj}_{\mathbf{a}}\mathbf{b} = (\mathbf{a}\cdot\mathbf{b})\mathbf{a}$.

3. The direction cosines of the vector $\mathbf{b}$ are:

   (a) $\cos(\alpha) = \frac{\mathbf{b}}{|\mathbf{b}|}\cdot\mathbf{i}$,

   (b) $\cos(\beta) = \frac{\mathbf{b}}{|\mathbf{b}|}\cdot\mathbf{j}$,

   (c) $\cos(\gamma) = \frac{\mathbf{b}}{|\mathbf{b}|}\cdot\mathbf{k}$,

   and so, α, β, γ are the respective angles that $\mathbf{b}$ makes with the x, y and z-axes.

Example 13 The vectors $\langle 1,2,-1\rangle$ and $\langle 3,-1,1\rangle$ are orthogonal, since $\langle 1,2,-1\rangle\cdot\langle 3,-1,1\rangle = 3-2-1 = 0$.

Example 14 Consider the vectors $\mathbf{a} = \langle 1,2,2\rangle$ and $\mathbf{b} = \langle 1,1,1\rangle$. Since $\text{proj}_{\mathbf{a}}\mathbf{b} = \frac{\mathbf{a}\cdot\mathbf{b}}{\mathbf{a}\cdot\mathbf{a}}\mathbf{a}$, then

$$\mathbf{v} = \mathbf{b} - \text{proj}_{\mathbf{a}}\mathbf{b} = \langle 1,1,1\rangle - \frac{5}{9}\langle 1,2,2\rangle = \langle \frac{4}{9}, -\frac{1}{9}, -\frac{1}{9}\rangle$$

must be perpendicular to $\mathbf{a}$ and must lie in the plane containing $\mathbf{a}$ and $\mathbf{b}$.

Definition 15 1. The **determinant** of the matrix M with rows vectors $\mathbf{v} = \langle a,b\rangle$ and $\mathbf{w} = \langle c,d\rangle$ can be calculated by: $|M| = \begin{vmatrix} a & b \\ c & d \end{vmatrix} = ad - bc$.

   The absolute value $|ad-bc|$ of this determinant equals the area of the parallelogram with sides $\mathbf{v}$ and $\mathbf{w}$.

2. The determinant of the matrix M with rows vectors $\mathbf{a} = \langle a_1,a_2,a_3\rangle$, $\mathbf{b} = \langle b_1,b_2,b_3\rangle$ and $\mathbf{c} = \langle c_1,c_2,c_3\rangle$ can be calculated by:

$$|M| = \begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix} = a_1\begin{vmatrix} b_2 & b_3 \\ c_2 & c_3 \end{vmatrix} - a_2\begin{vmatrix} b_1 & b_3 \\ c_1 & c_3 \end{vmatrix} + a_3\begin{vmatrix} b_1 & b_2 \\ c_1 & c_2 \end{vmatrix}$$

   The absolute value of the determinant $|M|$ equals the volume of the parallelepiped or box spanned by the vectors $\mathbf{a}$, $\mathbf{b}$ and $\mathbf{c}$.

3. The **cross product** $\mathbf{a}\times\mathbf{b}$ of vectors $\mathbf{a} = \langle a_1,a_2,a_3\rangle$ and $\mathbf{b} = \langle b_1,b_2,b_3\rangle$ can be calculated by:

$$\mathbf{a}\times\mathbf{b} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{vmatrix} = \begin{vmatrix} a_2 & a_3 \\ b_2 & b_3 \end{vmatrix}\mathbf{i} - \begin{vmatrix} a_1 & a_3 \\ b_1 & b_3 \end{vmatrix}\mathbf{j} + \begin{vmatrix} a_1 & a_2 \\ b_1 & b_2 \end{vmatrix}\mathbf{k}.$$

   The length of $\mathbf{a}\times\mathbf{b}$ is given by: $|\mathbf{a}\times\mathbf{b}| = |\mathbf{a}||\mathbf{b}|\sin(\theta)$, where $\theta\in[0,\pi]$ is the angle between $\mathbf{a}$ and $\mathbf{b}$. Also $|\mathbf{a}\times\mathbf{b}|$ is area of the parallelogram with sides $\mathbf{a}$ and $\mathbf{b}$. Note that it follows that area of the triangle with vertices $\langle 0,0,0\rangle$ and the position vectors $\mathbf{a}$ and $\mathbf{b}$ is $\frac{|\mathbf{a}\times\mathbf{b}|}{2}$.

Example 16 Consider the points $A = (1, 0, 1)$, $B = (0, 2, 3)$ and $C = (-1, -1, 0)$. Then the area of the triangle $\triangle$ with these vertices can be found by taking the area of the parallelogram spanned by $\overrightarrow{AB}$ and $\overrightarrow{AC}$ and dividing by 2. Thus:

$$\text{Area}(\triangle) = \frac{|\overrightarrow{\text{AB}} \times \overrightarrow{\text{AC}}|}{2} = \frac{1}{2}\left\| \begin{matrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ -1 & 2 & 2 \\ -2 & -1 & -1 \end{matrix} \right\| = \frac{1}{2}|\langle 0, -5, 5\rangle| = \frac{1}{2}\sqrt{0 + 25 + 25} = \frac{1}{2}\sqrt{50}$$

Example 17 Consider the vectors $\mathbf{a} = \langle 1, 0, 1\rangle$, $\mathbf{b} = \langle 0, 2, 3\rangle$ and $\mathbf{c} = \langle -1, 7, 0\rangle$. Then the volume of the parallelepiped or box spanned by these 3 vectors is:

$$\left\| \begin{matrix} 1 & 0 & 1 \\ 0 & 2 & 3 \\ -1 & 7 & 0 \end{matrix} \right\| = |-21 - 0 + 2| = |-19| = 19$$

Definition 18 If F is a force with magnitude A applied in the unit direction $\frac{\mathbf{a}}{|\mathbf{a}|}$ to an object in order to move it from the point P to the point Q, then the **work** W done is: $W = \frac{A}{|\mathbf{a}|}\mathbf{a} \cdot \overrightarrow{PQ}$.

Example 19 If F is a force of $10N$ (10 Newtons) applied in the unit direction $\frac{1}{\sqrt{6}}\langle 2, 1, 1\rangle$ to an object to move it from $P = (-3, -2, 5)$ to $Q = (1, 2, 3)$, then the work done is (length measured in meters):

$$W = \frac{10N}{\sqrt{6}}\langle 2, 1, 1\rangle \cdot \langle 4, 4, -2\rangle = \frac{100Nm}{\sqrt{6}},$$

where m is one meter.

Definition 20 The *torque* τ on a rigid body with position vector $\mathbf{a}$ with a force of magnitude A in the unit direction $\frac{\mathbf{b}}{|\mathbf{b}|}$ is:

$$\tau = \mathbf{a} \times A\frac{\mathbf{b}}{|\mathbf{b}|}.$$

Example 21 What is the magnitude (the length) of the torque on a rigid body with position vector $\mathbf{a} = \langle 1, -1, 3\rangle$ with a force of $10N$ in the direction of $\frac{\mathbf{b}}{|\mathbf{b}|} = \frac{1}{\sqrt{6}}\langle 2, 1, 1\rangle$ (length measured in meters m) ?

Solution :

$$|\tau| = |\langle 1, -1, 3\rangle \times \frac{10Nm}{\sqrt{6}}\langle 2, 1, 1\rangle| = |\frac{10Nm}{\sqrt{6}}\langle -4, 5, 3\rangle| = \frac{10Nm \cdot \sqrt{50}}{\sqrt{6}}.$$

Definition 22 Given a point $P = (x_0, y_0, z_0)$ and a vector $\mathbf{v} = \langle a, b, c\rangle$, the **vector equation** of the line L passing through P in the direction of $\mathbf{v}$ is:

$$\mathbf{r}(t) = P + t\mathbf{v} = \langle x_0, y_0, z_0\rangle + t\langle a, b, c\rangle = \langle x_0 + at, y_0 + bt, z_0 + ct\rangle.$$

The resulting equations:

$$x = x_0 + at,$$

$$y = y_0 + bt,$$

$$z = z_0 + ct,$$

are called the **parametric equations** for L. The resulting equations (solving for t):

$$\frac{x - x_0}{a} = \frac{y - y_0}{b} = \frac{z - z_0}{c},$$

are called the **symmetric equations** for L.

Example 23 The vector equations for the line L passing through $P = (1, 2, 3)$ and $Q = \langle 4, 0, 7 \rangle$ are given by:

$$\mathbf{r}(t) = P + t\overrightarrow{PQ} = \langle 1, 2, 3 \rangle + t\langle 3, -2, 4 \rangle = \langle 1 + 3t, 2 - 2t, 3 + 4t \rangle.$$

Definition 24 The plane passing through the point $P = (x_0, y_0, z_0)$ with normal vector $\mathbf{n} = \langle a, b, c \rangle$ is given by the following equation, where (x, y, z) denotes a general point on the plane:

$$0 = \mathbf{n} \cdot \langle x - x_0, y - y_0, z - z_0 \rangle.$$

Equivalently, we have:

$$a(x - x_0) + b(y - y_0) + c(z - z_0) = 0.$$

Example 25 The equation of the plane passing through $P = (1, 2, 3)$ and with normal vector $\mathbf{n} = \langle -3, 4, 1 \rangle$ is:

$$-3(x - 1) + 4(y - 2) + (z - 3) = 0.$$

Example 26 Find the equation of the plane passing through points $P = (1, 0, 2)$, $Q = (4, 2, 3)$, $R = (2, 0, 4)$.

Solution : Since a plane is determined by its normal vector $\mathbf{n}$ and a point on it, say the point P, it suffices to find $\mathbf{n}$. Note that:

$$\mathbf{n} = \overrightarrow{PQ} \times \overrightarrow{PR} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 3 & 2 & 1 \\ 1 & 0 & 2 \end{vmatrix} = \langle 4, -5, -2 \rangle.$$

So the equation of the plane is:

$$4(x - 1) - 5y - 2(z - 2) = 0.$$

Given two planes with unit normal vectors $\mathbf{n}_1$ and $\mathbf{n}_2$, respectively, then the cosine of the angle between them is the cosine of the angle between the lines determined by $\mathbf{n}_1$ and $\mathbf{n}_2$, which can be calculated using dot products.

Example 27 The cosine of the angle θ between $x - 2y + 2z = 1$ and $2x - y + 2z = 10$ is given by

$$\cos(\theta) = |\frac{1}{3}\langle 1, -2, 2 \rangle \cdot \frac{1}{3}\langle 2, -1, 2 \rangle| = \frac{1}{9}(2 + 2 + 4) = \frac{8}{9}.$$

Definition 28 Let $\mathbf{r}(t)$ be a vector valued curve in $\mathbb{R}^3$, where $\mathbf{r}(t) = \langle f(t), g(t), h(t)\rangle$. Here t is called the **parameter** of $\mathbf{r}(t)$. If the derivative $\mathbf{r}'(t) = \lim_{h\to 0} \frac{\mathbf{r}(t+h)-\mathbf{r}(t)}{h}$ exists for each t, then the curve $\mathbf{r}(t)$ is called differentiable and $\mathbf{r}'(t)$ is called the derivative or **velocity** or **tangent** vector field $\mathbf{v}(t) = \mathbf{r}'(t)$ to the curve $\mathbf{r}(t)$. The length $|\mathbf{v}(t)|$ is called the **speed** of the curve $\mathbf{r}$ at the parameter value t.

Theorem 29 If $\mathbf{r}(t) = \langle f(t), g(t), h(t)\rangle$ is a differential curve in $\mathbb{R}^3$, then:

$$\mathbf{r}'(t) = \langle f'(t), g'(t), h'(t)\rangle.$$

Conversely, if $f(t)$, $g(t)$, $h(t)$ are differentiable functions, then $\mathbf{r}(t)$ is differentiable. The speed function for $\mathbf{r}(t)$ is then:

$$\text{speed}(t) = \sqrt{(f'(t))^2 + (g'(t))^2 + (h'(t))^2}.$$

Example 30 Suppose $\mathbf{r}(t) = \langle t, \sin(2t), t^2+1\rangle$, then $\mathbf{r}'(t) = \langle 1, 2\cos(2t), 2t\rangle$ with $\mathbf{r}'(0) = \langle 1, 2, 0\rangle$. Hence, the tangent line to $\mathbf{r}(t)$ at $t = 0$ is given by:

$$L(t) = \mathbf{r}(0) + t\mathbf{r}'(0) = \langle 0, 0, 1\rangle + t\langle 1, 2, 0\rangle = \langle t, 2t, 1\rangle$$

and the speed function of $\mathbf{r}(t)$ is: $\text{speed}(t) = \sqrt{1 + 4\cos^2(2t) + 4t^2}$.

Definition 31 The length L of a parametrizing curve $\mathbf{r}(t)$ in $\mathbb{R}^3$ on a time interval $[a, b]$ is

$$L = \int_a^b |\mathbf{r}'(t)| dt.$$

Example 32 If $\mathbf{r}(t) = \langle \sin(t), \cos(t), 2t\rangle$, then $\mathbf{r}'(t) = \langle \cos(t), -\sin(t), 2\rangle$ with constant speed $\sqrt{\cos^2(t) + \sin^2(t) + 4} = \sqrt{5}$. Hence, the length of $\mathbf{r}(t)$ from time $t = 1$ to time $t = 6$ is:

$$L = \int_1^6 \sqrt{5}\, dt = \sqrt{5}t\Big|_1^6 = \sqrt{5}(6) - \sqrt{5}(1) = 5\sqrt{5}.$$

2 Some practice problems solved.

1. Find parametric equations for the line which contains $A(2, 0, 1)$ and $B(-1, 1, -1)$.
   Solution : Let $\mathbf{v} = \overrightarrow{AB} = \langle 2, 0, 1\rangle - \langle -1, 1, -1\rangle = \langle 3, -1, 2\rangle$. Since $A(2, 0, 1)$ lies on the line, then:
$$x = 2 + 3t,$$
$$y = 0 - t = -t,$$
$$z = 1 + 2t.$$

2. Determine whether the lines $l_1 : x = 1+2t$, $y = 3t$, $z = 2-t$ and $l_2 : x = -1+s$, $y = 4 + s$, $z = 1 + 3s$ are parallel, skew or intersecting.

   Solution : Vector part of line l_1 is $\mathbf{v}_1 = \langle 2, 3, -1\rangle$ and for line l_2 is $\mathbf{v}_2 = \langle 1, 1, 3\rangle$. Clearly, $\mathbf{v}_1$ is not a scalar multiple of $\mathbf{v}_2$ and so these lines are not parallel. If these lines intersect, then for some values of t and s:

$$x = 1 + 2t = -1 + s \;\Rightarrow\; 2t = -2 + s,$$

$$y = 3t = 4 + s \;\Rightarrow\; 3t = 4 + s.$$

Solving these two linear equations yields:

$$t = 6 \text{ and } s = 14.$$

Plugging these values into $z = 2 - t = 1 + 3s$ yields the inequality $-4 \neq 43$, which means there is no solution and the lines do not intersect. Thus, the lines are *skew*.

3. Find an equation of the plane which contains the points $P(-1, 2, 1)$, $Q(1, -2, 1)$ and $R(1, 1, -1)$.

   Solution : Consider the vectors $\overrightarrow{PQ} = \langle 2, -4, 0\rangle$ and $\overrightarrow{PR} = \langle 2, -1, -2\rangle$ which lie parallel to the plane. Then consider the normal vector:

$$\mathbf{n} = \overrightarrow{PQ} \times \overrightarrow{PR} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 2 & -4 & 0 \\ 2 & -1 & -2 \end{vmatrix} = 8\mathbf{i} + 4\mathbf{j} + 6\mathbf{k}.$$

   So the equation of the plane is given by:

$$\langle 8, 4, 6\rangle \cdot \langle x + 1, y - 2, z - 1\rangle = 8(x + 1) + 4(y - 2) + 6(z - 1) = 0.$$

4. Find the distance from the point $(1, 2, -1)$ to the plane $2x + y - 2z = 1$.

   Solution : The normal to the plane is $\mathbf{n} = \langle 2, 1, -2\rangle$ and the point $P = (0, 1, 0)$ lies on this plane. Consider the vector from P to $(1, 2, -1)$ which is $\mathbf{v} = \langle 1, 1, -1\rangle$. The distance from $(1, 2, -1)$ to the plane is equal to:

$$|\text{comp}_{\mathbf{n}}\mathbf{v}| = \left|\mathbf{v} \cdot \frac{\mathbf{n}}{|\mathbf{n}|}\right| = |\langle 1, 1, -1\rangle \cdot \frac{1}{3}\langle 2, 1, -2\rangle| = \frac{5}{3}.$$

5. Let two space curves

$$\mathbf{r}_1(t) = \langle \cos(t - 1),\, t^2 - 1,\, t^4\rangle, \quad \mathbf{r}_2(s) = \langle 1 + \ln s,\, s^2 - 2s + 1,\, s^2\rangle,$$

   be given where t and s are two independent real parameters. Find the cosine of the angle between the tangent vectors of the two curves at the intersection point $(1, 0, 1)$.

   Solution : After taking derivatives, we obtain:

$$\mathbf{r}_1'(t) = \langle -\sin(t - 1), 2t, 4t^3\rangle,$$

$$\mathbf{r}_2'(s) = \langle \frac{1}{s}, 2s - 2, 2s\rangle.$$

   At the point $(1, 0, 1)$, $t = 1$ and $s = 1$ and so, $\mathbf{r}_1'(1) = \langle 0, 2, 4\rangle$ and $\mathbf{r}_2'(1) = \langle 1, 0, 2\rangle$ are the related tangent vectors. Thus,

$$\cos(\theta) = \frac{\mathbf{r}_1'(1)}{|\mathbf{r}_1'(1)|} \cdot \frac{\mathbf{r}_2'(1)}{|\mathbf{r}_2'(1)|} = \frac{8}{\sqrt{20}\sqrt{5}} = \frac{4}{5}.$$

6. Suppose a particle moving in space has velocity

$$\mathbf{v}(t) = \langle \sin(t), \cos(2t), e^t \rangle$$

and initial position $\mathbf{r}(0) = \langle 1, 2, 0 \rangle$. Find the position vector function $\mathbf{r}(t)$.

Solution : Since $\mathbf{r}'(t) = \langle \sin(t), \cos(2t), e^t \rangle$, then $\mathbf{r}(t) = \int^t \mathbf{v}(s)ds$. Thus, $\mathbf{r}(t) = \langle -\cos(t) + x_0, \frac{1}{2}\sin(2t) + y_0, e^t + z_0 \rangle$ with $\mathbf{r}(0) = \langle -1 + x_0, y_0, 1 + z_0 \rangle = \langle 1, 2, 0 \rangle$. Thus, $x_0 = 2, y_0 = 2, z_0 = -1$ and so, $\mathbf{r}(t) = \langle -\cos(t) + 2, \frac{1}{2}\sin(2t) + 2, e^t - 1 \rangle$.

7. Find the center and radius of the sphere $x^2 + y^2 + z^2 + 6z = 16$.

Solution : Complete squares to obtain from $x^2 + y^2 + z^2 + 6z = 16$, the equation:

$$x^2 + y^2 + (z + 3)^2 = 16 + 9 = 25.$$

Hence, the center is at $C = (0, 0, -3)$ and the radius is 5.