Solution to Practice Final 3

1. a) $\overrightarrow{PQ} = \langle 2,-1,2\rangle - \langle 1,3,0\rangle = \langle 1,-4,2\rangle$ and $\overrightarrow{PR} = \langle 0,0,1\rangle - \langle 1,3,0\rangle = \langle -1,-3,1\rangle$. Thus, the normal vector to the plane is given by

$$\mathbf{n} = \overrightarrow{PQ} \times \overrightarrow{PR} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 1 & -4 & 2 \\ -1 & -3 & 1 \end{vmatrix} = \langle 2,-3,-7\rangle$$

and therefore the equation of the plane is given by $2(x-1) - 3(y-3) - 7z = 0$.

b) Substitute $\begin{cases} x = 2t^2 - 2 \\ y = t \\ z = 1 - t - t^2 \end{cases}$ in $x + y + z = 3$. Then, we have $t^2 = 4$, or $t = \pm 2$. Hence, the intersection points are $\begin{cases} (6,2,-5), \text{ when } t = 2 \\ (6,-2,-1), \text{ when } t = -2. \end{cases}$

2. By solving $\begin{cases} f_x(x,y) = 2x = 0 \\ f_y(x,y) = 4y - 2 = 0, \end{cases}$ we obtain a crtical point $(0,1/2)$ in the disk, and $f(0,1/2) = -1/2$.

Now, use Lagrange multiplier to find the max/min values on the boundary $x^2 + y^2 = 5$. We have to solve

$$\begin{cases} 2x = \lambda \cdot 2x \\ 4y - 2 = \lambda \cdot 2y. \end{cases}$$

From the first equation, we have $2x(1-\lambda) = 0$. Thus, $x = 0$ or $\lambda = 1$. If $x = 0$, we have $0^2 + y^2 = 5$, i.e. $y = \pm\sqrt{5}$ and we have

$$\begin{cases} f(0,\sqrt{5}) = 10 - 2\sqrt{5} \simeq 5.53 \\ f(0,-\sqrt{5}) = 10 + 2\sqrt{5} \simeq 14.47 \end{cases}$$

If $\lambda = 1$, from the second equation, we have $y = 1$. Then, by solving $x^2 + 1^2 = 5$, we have $x = \pm 2$, and $f(\pm 2, 1) = 4$. Therefore, we have

$$\begin{cases} \text{abs max} = 10 + 2\sqrt{5} \simeq 14.47 \text{ at } (0,-\sqrt{5}) \\ \text{abs min} = -1/2 \text{ at } (0,1/2). \end{cases}$$

(Note that, for this particular problem, we can substitute $x^2 = 5 - y^2$ in $f(x,y)$ and obtain $g(y) = y^2 - 2y + 5$ with $-\sqrt{5} \le y \le \sqrt{5}$. Then, we can obtain the max/min on the boundary by Calculus 1.)

3. $y = 2x$ and $y = x^2$ intersect at $(0,0)$ and $(2,4)$. Thus,

$$\iint_R xy\ dA = \int_0^2 \int_{x^2}^{2x} xy\ dy\, dx = \int_0^2 \left.\frac{xy^2}{2}\right|_{x^2}^{2x} dx$$
$$= \int_0^2 \left(2x^3 - \frac{x^5}{2}\right) dx = \left.\frac{x^4}{2} - \frac{x^6}{12}\right|_0^2 \simeq 2.67$$

(Note that we can also compute $\int_0^4 \int_{y/2}^{\sqrt{y}} xy\ dx\, dy$.)

4. a) Let $\begin{cases} P(x,y) = 2xy + \sin y \\ Q(x,y) = x^2 + x\cos y + 1 \end{cases}$. Then we have

$$\frac{\partial P}{\partial y} = 2x + \cos y = \frac{\partial Q}{\partial x}$$

Hence, $\mathbf{F}$ is conservative and there exists a function $f(x,y)$ such that

$$\begin{cases} f_x(x,y) = P(x,y) = 2xy + \sin y \\ f_y(x,y) = Q(x,y) = x^2 + x\cos y + 1 \end{cases}$$

By integrating the first in x, we have $f(x,y) = x^2y + x\sin y + g(y)$. Now, taking a partial derivative in y, we have $f_y(x,y) = x^2 + x\cos y + g'(y) = Q(x,y) = x^2 + x\cos y + 1$. i.e. $g'(y) = 1$. By integrating in y, we have $g(y) = y + C$.

Hence, we have found a potential function $f(x,y) = x^2y + x\sin y + y + C$, satisfying $\nabla f = \mathbf{F}$.

b) By the Fundamental Theorem of Calculus, we have

$$\int_C \mathbf{F} \cdot d\mathbf{r} = \int_C \nabla f \cdot d\mathbf{r} = f(2,4) - f(0,0) = 20 + 2\sin 4.$$

5. We can parametrize C by $\mathbf{r}(t) = \langle \cos t, \sin t \rangle$ with $0 \le t < 2\pi$. Then, we have

$$\oint_C \mathbf{F} \cdot d\mathbf{r} = \int_0^{2\pi} \mathbf{F}(\mathbf{r}(t)) \cdot \mathbf{r}'(t)\ dt$$
$$= \int_0^{2\pi} \langle \sin^2 t + \sin(\cos t), \cos t \sin t \rangle \cdot \langle -\sin t, \cos t \rangle\ dt$$
$$= \int_0^{2\pi} \big(-\sin^3 t - \sin(\cos t)\sin t + \cos^2 t \sin t \big)\ dt$$

With $-\sin^2 t = \cos^2 t - 1$, we have

$$= \int_0^{2\pi} \big(-\sin t - \sin(\cos t)\sin t + 2\cos^2 t \sin t \big)\ dt$$
$$= \left.\cos t - \cos(\cos t) - \frac{2}{3}\cos^3 t\right|_0^{2\pi} = 0.$$

(For the second and the third integrations, substitution was performed with $u = \cos t$.)

Note that we can also use Green's Theorem for this problem. Let D denote the unit disk. Then, we have

$$\begin{aligned}\oint_C \mathbf{F} \cdot d\mathbf{r} &= \iint_D \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \right) dA \\ &= \iint_D (-y)\ dA = 0.\end{aligned}$$

The last equality holds because the region D is symmetric in the y direction (and the integrand is just $-y$, an odd function.) Or just compute

$$\iint_D (-y)\ dA = \int_{-1}^{1} \int_{-\sqrt{1-x^2}}^{\sqrt{1-x^2}} (-y)\ dydx = 0.$$

6. Let C_1 be the (directed) line segment from $(0,0)$ to $(2,1)$ and C_2 be the (directed) line segment from $(2,1)$ to $(0,3)$. Then, we have

$$\int_C \mathbf{F} \cdot d\mathbf{r} = \int_{C_1} \mathbf{F} \cdot d\mathbf{r} + \int_{C_2} \mathbf{F} \cdot d\mathbf{r}.$$

Now, C_1 can be parametrized by $\mathbf{r}(t) = \langle 2t, t \rangle$ for $0 \le t \le 1$. Then, we have

$$\begin{aligned}\int_{C_1} \mathbf{F} \cdot d\mathbf{r} &= \int_0^1 \langle t^2, 2(2t)t + 2t \rangle \cdot \langle 2, 1 \rangle\ dt = \int_0^1 \left(6t^2 + 2t\right)\ dt \\ &= 2t^3 + t^2 \Big|_0^1 = 3.\end{aligned}$$

On the other hand, C_2 is along the line $y = -x + 3$. Thus, C_2 can be parametrized by $\mathbf{r}(t) = \langle t, -t+3 \rangle$ for $2 \ge t \ge 0$. Then, we have

$$\begin{aligned}\int_{C_2} \mathbf{F} \cdot d\mathbf{r} &= \int_2^0 \langle (-t+3)^2, 2t(-t+3) + t \rangle \cdot \langle 1, -1 \rangle\ dt = \int_2^0 \left(3t^2 - 13t + 9\right)\ dt \\ &= t^3 - 13\frac{t^2}{2} + 9t \Big|_2^0 = 0.\end{aligned}$$

Therefore, we have

$$\int_C \mathbf{F} \cdot d\mathbf{r} = 3.$$