Sketch of Solutions to Some Problems in Practice Final 1 & 2

Exam 1

1. Double integral. See no. 3 in Practice Exam 3.
2. Line integral. See no. 6 in Practice Exam 3.
3. Using the Polar coordinates, we have

$$R = \left\{(r,\theta) : 1 < r < \sqrt{2},\ 0 < \theta < \frac{\pi}{4}\right\}.$$

Then, we have (recall $y = r\sin\theta$ and $r^2 = x^2 + y^2$.)

$$\iint_R y\sqrt{x^2+y^2}\,dA = \int_0^{\pi/4}\int_1^{\sqrt{2}} r\sin\theta\sqrt{r^2}\,r\,dr\,d\theta = \left(-\cos\theta\Big|_0^{\pi/4}\right)\left(\frac{r^4}{4}\Big|_1^{\sqrt{2}}\right).$$

4. Conservative vector field, finding its potential, the Fundamental Theorem of Calculus. See no. 4 in Practice Exam 3.
5. a) The normal vector to the level surface at $(1,1,-1)$ is given by

$$\nabla f(x,y,z)\Big|_{(1,1,-1)} = \langle 6x+yz, xz, xy+3z^2\rangle\Big|_{(1,1,-1)} = \langle 5,-1,4\rangle.$$

Thus, the equation of the tangent plane to the level surface at $(1,1,-1)$is given by

$$5(x-1) - (y-1) + 4(z+1) = 0.$$

b) We have

$$\mathbf{r}'(t)\Big|_{t=1} = \langle 4t-1, -2t^{-3}, 2t-6t\rangle\Big|_{t=1} = \langle 3,-2,-4\rangle.$$

The unit vector $\overrightarrow{u}$ pointing in the same direction is given by $\overrightarrow{u} = \frac{1}{\sqrt{29}}\langle 3,-2,-4\rangle$. Then, we have

$$\mathrm{D}_{\overrightarrow{u}} f(1,1,-1) = \nabla f(1,1,-1)\cdot\overrightarrow{u} = \langle 5,-1,4\rangle\cdot\frac{1}{\sqrt{29}}\langle 3,-2,-4\rangle.$$

6. We have

$$\begin{cases} f_x(x,y) = 2x - 2y = 0 \\ f_y(x,y) = -2x + 4y - 2 = 0 \end{cases}$$

From the first equation, we have $x = y$. Substituting $x = y$ in the second, we have $y = 1$ and thus $x = y = 1$. And, we have$f(1, 1) = -1$.

Now, divide the boundary into 3 line segments, l_1, l_2, l_3, where l_1 is the part of the x-axis, l_2 is the part of the y-axis, and l_3 is the part of the $x + y = 7$.

On l_1 (i.e. $y = 0$ and $0 \leq x \leq 7$), we have $f(x, 0) = x^2$. Thus, we have max $= 49$ at $(7, 0)$ and min $= 0$ at $(0, 0)$.

On l_2 (i.e. $x = 0$ and $0 \leq y \leq 7$), we have $g(y) = f(0, y) = 2y^2 - 2y$. Since $g'(y) = 4y - 2 = 0$ gives $y = 1/2$, we have max $= 84$ at $(0, 7)$ and min $= -1/2$ at $(0, 1/2)$. ($f(0, 0) = 0$ and this is neither max nor min.)

On l_3 (i.e. $x = 7 - y$ with $0 \leq y \leq 7$), we have $g(y) = f(7 - y, y) = 5y^2 - 30y + 49$. Since $g'(y) = 10y - 30 = 0$ gives $y = 3$, we have max $= 84$ at $(0, 7)$ and min $= 4$ at $(4, 3)$. ($f(0, 7) = 49$ and this is neither max nor min.)

Hence, abs max $= 84$ at $(7, 0)$ and abs min $= -1$ at $(1, 1)$.

__Exam 2__

1. a) We have $\nabla f(x, y)\Big|_{(1,0)} = \langle e^{xy} + xye^{xy}, x^2 e^{xy}\rangle\Big|_{(1,0)} = \langle 1, 1\rangle$.
Since the direction is given by $\langle 3, 2\rangle - \langle 1, 0\rangle = \langle 2, 2\rangle$, the unit vector in the same direction is given by $\overrightarrow{u} = \frac{1}{\sqrt{8}}\langle 2, 2\rangle$ Hence, we have

$$\mathrm{D}_{\overrightarrow{u}} f(1, 0) = \langle 1, 1\rangle \cdot \frac{1}{\sqrt{8}}\langle 2, 2\rangle.$$

b) Let $\overrightarrow{u} = \langle a, b\rangle$ with $a^2 + b^2 = 1$. Since $0 = \mathrm{D}_{\overrightarrow{u}} f(1, 0) = \langle 1, 1\rangle \cdot \langle a, b\rangle = a + b$, we have $2a^2 = 1$. Hence $\overrightarrow{u} = \langle \pm\frac{1}{\sqrt{2}}, \mp\frac{1}{\sqrt{2}}\rangle$.

2. a) Let C be the counterclockwise unit circle, and D be the unit disk. Then, by Green's theorem, the work is given by

$$\oint_C \mathbf{F} \cdot d\mathbf{r} = \iint_D \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) dA = \iint_D 2\ dA$$
$$= 2 \cdot (\text{Area of the unit disk}) = 2\pi.$$

b) By Greene's theorem, we have

$$\oint_C (-y^2)\, dx + xy\, dy = \int_0^1 \int_0^1 \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) dxdy = \int_0^1 \int_0^1 3y\ dxdy$$
$$= \left(x\Big|_0^1\right)\left(\frac{3}{2}y^2\Big|_0^1\right).$$

3. Conservative vector field, finding its potential, the Fundamental Theorem of Calculus. See no. 4 in Practice Exam 3.

4.

a) Equation of the tangent plane to a level surface at a given point. See no. 5 in Practice Exam 1.

b) The direction vector is given by $\nabla F(1,2,-2\sqrt{5})$, where $F(x,y,z) = x^2 + y^2 - \frac{1}{4}z^2$. Then, the equation of the normal line is given by

$$\mathbf{r}(t) = \langle 1, 2, -2\sqrt{5}\rangle + t \cdot \nabla F(1,2,-2\sqrt{5}).$$

5. $\begin{cases} \text{Inside: finding critical points and evaluating the function at those points} \\ \text{On the boundary: Lagrange multiplier.} \end{cases}$

See no. 2 in Practice Exam 3.

6. Integrating $\mathbf{a}(t)$, we have $\mathbf{v}(t) = \langle -3\sin t + C_1, 3\cos t + C_2, 2t + C_3\rangle$. By evaluating at $t = 0$, we have $\langle 0, 3, 0\rangle = \mathbf{v}(0) = \langle C_1, 3 + C_2, C_3\rangle$ i.e. $C_1 = C_2 = C_3 = 0$.

If $\mathbf{v}(t)$ is perpendicular to $\mathbf{a}(t)$, then we have $\mathbf{v}(t) \cdot \mathbf{a}(t) = 0$.

$$\text{i.e. } \mathbf{v}(t) \cdot \mathbf{a}(t) = 9\sin t\cos t - 9\cos t\sin t + 4t = 4t = 0.$$

Therefore, $t = 0$.

7. The direction vector $\mathbf{v}$ of the line is given by

$$\mathbf{v} = \langle 3, -6, -2\rangle \times \langle 2, 1, -2\rangle = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 3 & -6 & -2 \\ 2 & 1 & -2 \end{vmatrix} = \langle 14, 2, 15\rangle.$$

Now, to find a point on the line of the intersection of the two planes, substitute $z = 0$. Then, we have

$$\begin{cases} 3x - 6y = 15 \\ 2x + y = 5. \end{cases}$$

By sloving, we obtain $(3, -1, 0)$. Hence the equation of the line is given by

$$\mathbf{r}(t) = \langle 3, -1, 0\rangle + t\langle 14, 2, 15\rangle.$$