DEPARTMENT OF MATHEMATICS AND STATISTICS
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MATH 233 **Solution to EXAM 1** **Fall 2010**

1. Given two vectors $\mathbf{a} = \mathbf{i} + \mathbf{j} + \mathbf{k}$, $\mathbf{b} = \mathbf{i} - \mathbf{j} + \mathbf{k}$

(a) Find $\mathbf{proj_a b}$, the vector projection of $\mathbf{b}$ onto $\mathbf{a}$.

First calculate $\mathbf{comp_a b}$:

$$\mathbf{comp_a b} = \frac{\mathbf{a} \cdot \mathbf{b}}{|\mathbf{a}|} = \frac{1}{\sqrt{3}}.$$

So $\mathbf{proj_a b} = \mathbf{comp_a b} \frac{\mathbf{a}}{|\mathbf{a}|} = \frac{1}{\sqrt{3}} \frac{\langle 1,1,1 \rangle}{\sqrt{3}} = \left\langle \frac{1}{3}, \frac{1}{3}, \frac{1}{3} \right\rangle$

(b) Find the angle θ formed by $\mathbf{a}$ and $\mathbf{b}$.

Use formula $\cos\theta = \frac{\mathbf{a} \cdot \mathbf{b}}{|\mathbf{a}||\mathbf{b}|}$. We get $\cos\theta = \frac{1}{3}$
and $\theta = \arccos(1/3)$.

2. Consider the points $P(2, 2, 6)$, $Q(0, 5, 5)$, $R(3, 1, 7)$.

(a) Find a nonzero vector orthogonal to the plane through the points P, Q, and R.

$$\overrightarrow{PQ} = \langle -2, 3, -1 \rangle$$

$$\overrightarrow{PR} = \langle 1, -1, 1 \rangle$$

$$\overrightarrow{PQ} \times \overrightarrow{PR} = \begin{bmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ -2 & 3 & -1 \\ 1 & -1 & 1 \end{bmatrix} = 2\mathbf{i} + \mathbf{j} - \mathbf{k}$$

The vector $\overrightarrow{PQ} \times \overrightarrow{PR}$ is
orthogonal to the plane through the points P, Q, and R.

(b) Find the area of the triangle PQR.

We know $\left|\overrightarrow{PQ} \times \overrightarrow{PR}\right|$ is the
area of the parallelogram spanned by the vectors
$\overrightarrow{PQ}$ and $\overrightarrow{PR}$, therefore,

$$A_{\Delta PQR} = \frac{|\overrightarrow{PQ} \times \overrightarrow{PR}|}{2} = \frac{\sqrt{2^2 + 1^2 + (-1)^2}}{2} = \frac{\sqrt{6}}{2}$$

3. Find an equation of the plane that passes through the point $(9, 0, -3)$ and contains the given line $x = 7 - 2t$, $y = 1 + 3t$, $z = 6 + 4t$.

To find a normal vector $\mathbf{n}$ to the plane, we will first find two non-parallel vectors $\mathbf{v}_1$ and $\mathbf{v}_2$ on the plane. We can take $\mathbf{n} = \mathbf{v}_1 \times \mathbf{v}_2$ which will be orthogonal to both $\mathbf{v}_1$ and $\mathbf{v}_2$ and thus normal to the plane.

Since the given line lies on the plane, its direction vector $\langle -2, 3, 4\rangle$ is on the plane. Let $\mathbf{v}_1 := \langle -2, 3, 4\rangle$.

To find the second vector we look at two points on the plane, one of which must not be on the line. We can take our two points to be $P_1\,(9, 0, -3)$ and $P_2\,(7, 1, 6)$, the former being the given point on the plane and the latter being the point on the line at $t = 0$. Then $\mathbf{v}_2 := \langle -2, 1, 9\rangle$ is the vector going from P_1 to P_2.

$$\begin{aligned}
\mathbf{n} &= \mathbf{v}_1 \times \mathbf{v}_2 \\
&= \langle -2, 3, 4\rangle \times \langle -2, 1, 9\rangle \\
&= \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ -2 & 3 & 4 \\ -2 & 1 & 9 \end{vmatrix} \\
&= \langle 23, 10, 4\rangle
\end{aligned}$$

Using the normal vector $\mathbf{n}$ and the point $P_1\,(9, 0, -3)$ on the plane, the equation of the plane is given by:

$$23(x - 9) + 10(y - 0) + 4(z + 3) = 0$$

Which can be simplified to:

$$23x + 10y + 4z - 195 = 0$$

4. We all know that $x^2 + y^2 + z^2 = 26$ is a sphere, denoted as S, in the space.

   (a) Is any of the surfaces $x - y^2 = z^2$, $x^2 + \frac{y}{5}^2 = 1 - z^2$ or $x^2 + y^2 - z^2 = 24$ inside of that sphere S?

   The surface $x - y^2 = z^2$ is not in the sphere; rewriting it as $x = y^2 + z^2$, one sees that it is a parabaloid and thus unbounded. The surface $x^2 + \frac{y^2}{5} + z^2 = 1$ is an ellipsoid. It's extreme points on the three axes are $(\pm 1, 0, 0)$, $(0, \pm\sqrt{5}, 0)$ and $(0, 0, 1)$, all of which are inside the sphere. Thus the entire ellipsoid is inside the sphere.
   The surface $x^2 + y^2 - z^2 = 24$ is a hyperbaloid of one sheet, and thus is unbounded, so it cannot be inside the sphere.

   (b) What type of curve is the intersection of $z = x^2 + y^2$ with that sphere S?

   It is a circle, as one sees easily from graphing the two surfaces.

   (c) Find the points of intersection between the helix $\langle \cos t, \sin t, t \rangle$ and that sphere S.

   Replace the curve in the equation

$$(\cos t)^2 + (\sin t)^2 + t^2 = 26$$

   then $t = -5, 5$. So it intersects at the points $(\cos(-5), \sin(-5), -5)$ and $(\cos(5), \sin(5), 5)$.

5. Find a parametric equation that represents the curve of intersection of the two surfaces. The cylinder $x^2 + y^2 = 25$ and the surface $z = xy$

The projection of the cylinder onto the xy-plane is the circle $x^2 + y^2 = 25$, $z = 0$. So we can parametrize it by:

$$x = 5\cos t, \quad y = 5\sin t; \quad 0 \le t \le 2\pi.$$

Now from the equation of the surface $z = xy$ we have

$$z = (5\cos t)(5\sin t) = 25\cos t \sin t.$$

So we can write the parametric equation for the curve of intersection C as:

$$x = 5\cos t, \quad y = 5\sin t, \quad z = 25\cos t \sin t; \quad 0 \le t \le 2\pi.$$

6. Find $\mathbf{r}(t)$ if $\mathbf{r}'(t) = 8t^7\mathbf{i} + 4t^3\mathbf{j} + \sqrt{t}\mathbf{k}$ and $\mathbf{r}(1) = \mathbf{i} + \mathbf{j}$.

Since $\mathbf{r}'(t) = 8t^7\mathbf{i} + 4t^3\mathbf{j} + \sqrt{t}\mathbf{k}$, we know that $\mathbf{r}(t)$ will be some antiderivative of this function, so

$$\mathbf{r}(t) = \left\langle t^8, t^4, \frac{2}{3}t^{3/2} \right\rangle + \langle C_1, C_2, C_3 \rangle .$$

Using this formula, we see that $\mathbf{r}(1) = \left\langle 1 + C_1, 1 + C_2, \frac{2}{3} + C_3 \right\rangle$, so since $\mathbf{r}(1) = \langle 1, 1, 0 \rangle$, we can solve for C_1, C_2, C_3 to get $C_1 = 0$, $C_2 = 0$, and $C_3 = -\frac{2}{3}$. Thus:

$$\mathbf{r}(t) = \left\langle t^8, t^4, \frac{2}{3}t^{3/2} - \frac{2}{3} \right\rangle .$$

7. The position function of a particle is $\mathbf{r}(t) = \langle t^2, 5t, t^2 - 16t \rangle$. When is the speed of the particle a minimum?

First, find the velocity:

$$\mathbf{v}(t) = \mathbf{r}'(t) = \langle 2t, 5, 2t - 16 \rangle$$

Then the speed is

$$|\mathbf{v}(t)| = \sqrt{4t^2 + 25 + (2t - 16)^2} = \sqrt{8t^2 - 64t + 281}$$

The minimum of the speed occurs when the function $g(t) = 8t^2 - 64t + 281$ has a minimum. Since $g'(t) = 16t - 64$ has its only zero at $t = 4$, and since
$g''(4) = 16 > 0$,
we know that when $t = 4$, the speed of the particle is indeed a minimum.