Ergodicity and Coexistence of elliptic islands in a family of convex billiards

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In this paper we study a two-parameter family of convex billiard tables, by taking the intersection of two round disks (with different radii) in the plane. These tables give a generalization of the one-parameter family of lemon-shaped billiards. Initially, there is only one ergodic table among all lemon tables. In our generalized family, we observe numerically the prevalence of ergodicity among the some perturbations of that table. Moreover, numerical estimates of the mixing rate of the billiard dynamics on some ergodic tables are also provided.

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The billiard dynamics are determined by the shapes of the tables, and may vary greatly from completely integrable to strongly chaotic behaviors. In spite of its potential appeal to a broad range of physical applications, the hyperbolicity has remained a difficult problem to prove for general billiards. The “defocusing mechanism”, firstly discovered by Bunimovich, states that, if the boundary of a convex table is made of circular arcs, such that any circle that contains one circular arc boundary component lies entirely inside the table, then the billiard system is hyperbolic. This mechanism is further generalized to absolutely focusing boundary components, after various improvements. There are many convex billiards that fail the defocusing mechanism, and only a few of which are known to be fully chaotic. In this paper we found numerically a new class of convex billiard tables $Q(R, B)$ with ergodic properties. They are simple tables obtained by intersection of a unit disk with another disk of radius $R \geq 1$ such that the centers of these two disks are $B > 0$ units away. On one hand, these tables belong to the simplest type that fails the Bunimovich’s defocusing mechanism, but may still enjoy rich ergodic and chaotic properties. On the other hand, the boundary of our billiard table consists only two circular arcs, which makes the billiard systems much easier to study comparing to other billiards, say the oval tables and squash tables. Such systems are also very important to physicists since they exemplify a delicate transition from the regular behavior to chaos. A better understanding of this type of billiards will contribute to uncover the nature of hyperbolicity for general billiards systems.

I. INTRODUCTION

Billiard systems are a class of dynamical systems originating in statistical mechanics, in which a particle moves freely along strict segments in a bounded region in the plane (which is called the billiard table), and changes its velocity according to the law of elastic reflection upon collisions with the boundary of the billiard table. The dynamics of the billiard systems are determined by the shapes of the tables, and may vary greatly from regular (completely integrable) to strongly chaotic behaviors.
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The study of billiard systems was pioneered by Sinaǐ in his seminal paper on dispersing billiards, where he proved the hyperbolicity and ergodicity of these systems and derived various statistical properties. The mechanism for the hyperbolicity of dispersing billiards is that, dispersing wave-fronts remain dispersing after each collision. In 1974 Bunimovich discovered the defocusing mechanism of convex billiard tables, and proved the hyperbolicity and ergodicity of stadium billiards. So a convex table may also be hyperbolic if the focusing wavefronts spend enough time on defocusing. See also [6, 11, 19, and 24] for various improvements of the defocusing mechanism and new ergodic tables.

There are only a few model of billiards that fail this mechanism and are known to be fully chaotic. In [4] Benettin and Strelcyn introduced the oval tables and observed the bifurcation phenomena, the coexistence of elliptic and chaotic regions, and the separation of the chaotic region into several invariant components. Moreover they gave numerical estimates of the Lyapunov exponent and the entropy of these billiard dynamical systems. In [3] the authors studied a two-parameter family of convex billiard tables, the squash billiard tables, on which the defocusing mechanism does not take place. They gave a numerical and a heuristic proof of the ergodicity of squash billiards. For more related discussions see [13, 15–17].

Another family of convex tables, the lemon shaped tables, was introduced by Heller and Tomsovic in 1993, by taking the intersection of two unit disks. The coexistence of the elliptic islands and chaotic region has also been observed numerically in [18 and 21] for most lemon tables. The only possible exception is when the centers lie on each others’ boundaries, which is the starting point of our study. In fact, we put the lemon tables under a more general family, among which the ergodicity may no longer be an exceptional phenomenon. Our tables are also simple, obtained by intersection of a unit disk $D_1$ with another round disk $D_R$ with radius $R \geq 1$, see Fig. 1, where $B$ measures the distance between the centers of these two disks. On one hand, the boundary of our billiard table consists of two circular arcs, which makes the billiard systems much easier to study. Yet on the other hand, these systems already exhibit rich dynamical behaviors. We have found that there exists an infinite strip $\mathcal{D} \subset [1, \infty) \times [0, \infty)$, such that for any $(R, B) \in \mathcal{D}$, the billiards on $Q(R, B)$ is ergodic.

Taking the degenerate case when $R = 1$ and $B = 0$, one can check that $Q(1, 0)$ is a unit disk table, which is completely integrable. On the other hand, letting $R \to \infty$, the limit cases are various tables obtained by cutting a disk by a straight line (see also [21]). It is well
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FIG. 1: Basic construction of our billiard table $Q(R, B)$

known that the resulting table is hyperbolic and ergodic if and only if the curved boundary component is a major-arc. Note that the billiard table is degenerate or empty if $B \geq R + 1$, as the two disks do not intersect each other. Similarly, $Q(R, B) = D_1$ if $B \leq R - 1$, since $D_1 \subset D_R$. In the second case, there are also interesting dynamics if we set the billiard table to be the annulus between the two disks: $A(R, B) = D_R \setminus D_1$ (when $R > B + 1$). In fact, this annulus table has already been studied extensively\textsuperscript{2,12,22} from the 80’s. So in this paper, we will focus on the family of tables $Q(R, B)$ with the parameter space

$$\Omega = \{(R, B) \subset (1, \infty) \times (0, \infty) : -1 < B - R < +1\},$$

which contains new billiard tables that have never been studied before. Fig. 2 clarifies the regions in the first quarter $(R, B) \in [0, \infty) \times [0, \infty)$, where $\Omega' = \{(R, B) \subset [0, 1] \times (0, \infty) : -1 < B - R < 1\}$ refers to an equivalent class of billiards as those in $\Omega$ (by switching the roles of $r$ and $R$). There are three regions in $[0, \infty) \times [0, \infty)$: $B > R + 1$ refers to the degenerate case; for $B < R - 1$, also degenerate, and interesting dynamics happens in the annulus table; $\Omega'$ and $\Omega$ refer to two equivalent families of 2-parameter convex billiards. The subregions I,II and III in $\Omega$ are characterized by the relative positions of the two centers with respect to the table $Q(R, B)$.

It is observed in [22] that the phase space of annulus billiard $A(R, B) = D_R \setminus D_1$ is divided into three subregions (when $0 < B < R - 1$): the completely integrable region in which the billiard trajectories never hit the inner circle; the nearly integrable region in which the billiard trajectories have strictly alternative collisions between the inner and outer circles; the chaotic region in which the collisions of the trajectories ‘randomly’ alternate with the inner and outer circles. Recently, Bunimovich\textsuperscript{7} constructed the first class of natural and visible systems
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with coexistence phenomena: mushroom billiards, which combine the completely integrable dynamics on elliptic table and the completely chaotic dynamics on elliptic stadium. In [18 and 21], the authors also observed numerically the bifurcations and coexistence of chaotic regions and elliptic islands on most lemon billiards, when the distance $B$ is either too large ($B > 1$) or too small ($B < 1$). Similar phenomena also appear in our tables $Q(R, B)$ for $R > 1$, when $B$ is either too large ($B > R$) or too small ($B < 1$). So our billiard systems also supply the simple examples that exhibiting the coexistence phenomenon.

Now we conclude the following observation from our various numerical simulations:

**Observation.** For parameters in a larger set of Region I, the billiard tables $Q(R, B)$ are ergodic.

This is demonstrated in Fig. 3. The small solid region contains parameters with which the billiards have elliptic island, and the parameters undergoes deformations and disappear when the parameters leave the solid region. The rest of parameters in the shaded Region I corresponding to tables with ergodic property (according to our simulation). The dotted curve in Region I has equation $B = \sqrt{R^2 - 1}$. The segment over $R = 1$ corresponds to the one-parameter family lemon tables.

Intuitively, our observation says that, the billiard system is likely to be ergodic if the distance between the two centers takes the intermediate values $1 \leq B \leq R$. Note that the limit parameter $B \leq R \to \infty$ corresponds to the major-arc table, which is known to be hyperbolic and ergodic. Moreover, we also propose the following conjecture:
Observation. There exists $R_0 \gg 1$, such that for all $R > R_0$, billiards on the table $Q(R, B)$ is ergodic provided that $B \in (1, \sqrt{R^2 - 1})$.

The geometric meaning of the condition $B \in (1, \sqrt{R^2 - 1})$ is that the boundary of the table contains a major arc. In fact, the tables in Section III C can also be viewed as small perturbations of the major-arc table: the table obtained by simply closing a major arc by a straight line segment $AD$. Denote such a table by $Q_0$. It is well known that the billiard dynamics on $Q_0$ is equivalent to that of a table with a boundary consisting of two identical major arcs and hence satisfies Bunimovich’s defocusing mechanism. So the dynamics on the table $Q_0$ is hyperbolic and ergodic. We then alter the curvature of the curve connecting $A$ and $D$ by varying the radius $R$ (then the center distance $B$ changes accordingly). Although an arbitrary small perturbation can make the table fail the defocusing mechanism, but our simulation shows that the ergodicity may survive under these small perturbations, as long as the table continues to satisfy the condition $1 \ll B < R$.

FIG. 3: Red region: there exists an elliptic period-3 orbit for each table in this region, and we observe a small elliptic island surrounding it, which clearly destroys the ergodicity.

The paper is organized as follows: in Section 2 we give a brief introduction of general billiard systems and some features of our billiard table $Q(R, B)$. In Section III we first study a special table $Q(1, 1)$ and verify its ergodicity numerically. Then we examine three different types of perturbations of the billiard table $Q(1, 1)$ with the three parameters satisfying $1 < B < R$ (Region I), $1 < B = R$ (the boundary of Region I along the diagonal), and
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$1 = B < R$ (the boundary at the bottom of Region I). We observe the ergodicity of billiard tables with parameters in a large set in these family, and also detect tables with a small region of parameters among which the ergodicity fails due to the existence of elliptic periodic orbits. In each subsection we examine the dominating periodic orbits and their effects on the dynamics. In Section 5 we study the non-ergodic perturbations in Region II and III and observe the bifurcation of periodic orbits and the generation of elliptic islands surrounding them. In Fig. 3 we summarize the conclusions obtained in this study.

Although parts of our results are only based on numerical simulations, a rigorously mathematical justification is currently under investigation. The most difficult step is to prove hyperbolicity, especially since the classical defocusing mechanics fails in our model, and it is not obvious that the hyperbolicity can be guaranteed by considering any fixed higher iterations. Instead, our preliminary calculation shows that one should define a stopping time function $\tau$ and the associated induced map $F(x) = T^{\tau(x)}x$. By properly chosen the stopping time it is possible that the induced map enjoys hyperbolicity as well as ergodicity.

II. PRELIMINARIES

In this section we first introduce the notations of billiard systems and then describe some basic properties of our billiard table. Let $Q$ be a compact convex domain in the plane, $\Gamma = \partial Q$ be the boundary of $Q$ equipped with the arc-length parametrization. The phase space of the billiard system on $Q$ is a cylinder $M = \Gamma \times [-\pi/2, \pi/2]$. A point $x \in M$ has the coordinate representation $x = (s, \varphi)$, where $s$ is measured by its arc-length along the oriented boundary $\Gamma$, and $\varphi$ is the angle measured from the inner normal direction to the outgoing velocity vector after the reflection. The billiard map $T : M \rightarrow M$ sends a point $(s, \varphi)$ to the point $(s_1, \varphi_1)$ right after its next collision with $\Gamma$. The derivative $DT$ at the point $x = (s, \varphi)$ is denotes as $D_{(s,\varphi)}T$, which is given by (see [8, (2.26)]):

$$
\begin{pmatrix}
\tau K + \cos \varphi \\
\tau K K_1 + K \cos \varphi_1 + K_1 \cos \varphi \\
\tau K_1 + \cos \varphi_1
\end{pmatrix}
$$

where $(s_1, \varphi_1) = T(s, \varphi)$, $K$ is the curvature of radius of $\Gamma$ at $\Gamma(s)$, and $K_1$ is the curvature of radius of $\Gamma$ at $\Gamma(s_1)$. Moreover, $T$ preserves a natural measure $d\mu = c \cdot \cos \varphi \, ds \, d\varphi$, where $c$ is a normalizing constant. See [8] for more information.
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Now we consider a 2-parameter billiard table $Q(R, B)$, obtained by intersecting a unit disk $D_1$ with a larger disk $D_R$ of radius $R$, with the distance $B$ between their centers. As noted in the introduction, we always assume that $R > 1$ and $R - 1 < B < R + 1$. There are two corners on the table, which break down the smoothness of the boundary $\Gamma$ and will lead to the existence of nontrivial singularity curves. More precisely, the singularity set of this table $Q(R, B)$ consists of two vertical segment in $M$ based at these two corner points, as well as the horizontal lines $\varphi = \pm \pi/2$.

Note that for the billiard dynamics, $\det(D_x T) = \cos \varphi$ and hence $\det(D_x T^k) = 1$ for all periodic orbits $x = T^k x$. Moreover, a periodic point $x = T^k x$ is said to be hyperbolic, parabolic and elliptic, if $|\text{Tr}(D_x T^k)| > 2$, $|\text{Tr}(D_x T^k)| = 2$ and $|\text{Tr}(D_x T^k)| < 2$, respectively. It is easy to see that on each table $Q(R, B)$, there exists exactly one new periodic orbit $O_2$ of period 2 hitting both arcs, which bounces perpendicularly between the midpoints $\{p_1, p_2\}$ of the circle arcs (see Fig. 4). Other period-2 orbits, if exist, hit only the arc of the unit disk. The coordinate representation of this period-2 orbit is given by $O_2 = \{(p_1, 0), (p_2, 0)\}$.

Proposition II.1. Let $O_2$ be the period-2 orbit of the billiard map on the table $Q(R, B)$. Then this orbit is hyperbolic if $1 < B < R$, is parabolic if $B = 1$ or $R = B$, and elliptic if $B < 1$ or $B > R$.

This proposition is proved by Wojtkowski. A proof in included here for completeness.

Proof. Note that the travel time $\tau$ between the collisions satisfies $\tau + B = 1 + R$, $K_{p_1} = -1$ and $K_{p_2} = -1/R$. Then after a simple calculation, the trace of the derivative $D_{(p_1, 0)} T^2$ is
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(by Eq. (1))

$$\text{Tr}(D_{(p_1,0)}T^2) = 4(1 - \tau)(1 - \tau/R) - 2$$

$$= 4 \left(1 - \frac{B}{R}\right)(1 - B) - 2.$$ 

So there are three different qualitative behaviors of the periodic orbits:

1. If $1 < B < R$, then $\text{Tr}(D_{(p_1,0)}T^2) < -2$, hence \{(p_1, 0), (p_2, 0)\} is a hyperbolic periodic orbit.

2. If $B = 1$ or $B = R$, then $\text{Tr}(D_{(p_1,0)}T^2) = -2$, hence \{(p_1, 0), (p_2, 0)\} is a parabolic orbit.

3. If $R - 1 < B < 1$ or $R < B < R + 1$, then $-2 < \text{Tr}(D_{(p_1,0)}T^2) < 2$, hence \{(p_1, 0), (p_2, 0)\} is an elliptic orbit.

This finishes the proof of the proposition. \qed

III. ERGODIC TABLES IN REGION I

In this section we will investigate the ergodicity of tables with parameters in Region I. From Fig. 2 we know that Region I is an unbounded strip with three line boundary components. Clearly, for parameters on the line segment $B = R - 1$, $Q(R, B)$ represents the unit disk $D_1$ and the dynamics is completely integrable. To investigate the boundary $1 = B < R$ as well as $1 < B = R$, we first start with a special case of our two-parameter family, the table with $B = R = 1$. This table $Q(1, 1)$ has been well studied (see [18, Fig. 4(f)] after setting their parameter $w = 0.5$, and also in [21, Fig. 4(e)] after setting their parameter $\delta = \delta_c$). In particular, it is observed that this table is indeed ergodic by numerical simulations (see also Fig. 5, in which we demonstrate the iterations along one typical phase point (1.5, 0.01) after 100,000, 1,000,000 and 100,000,000 iterations, respectively. Our numerical results show that, visually, the phase space is completely filled after 100,000,000 iterations. Recall that a simple criterion for a measure-preserving system $(X, \mu, T)$ to be ergodic is, the averages $\frac{1}{n} \sum_{k=0}^{n-1} \delta_{T^kx}$ has the same limit distribution for $\mu$-a.e. $x \in X$. We also tried quite a few of different initial points and get the same asymptotic distribution for large enough iterations. So the billiard dynamics on $Q(1,1)$ should be ergodic.
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FIG. 5: Trajectory segments with an initial point (1.5, 0.01) on the phase space of the table $Q(1, 1)$, after 100,000, 1,000,000 and 100,000,000 iterations, respectively.

To get a better understanding of the dynamical systems, it is often rewarding to study its statistical properties. The starting point of this investigation is the decay of correlations. Recall that the billiard map $T$ preserves a natural measure $d\mu = c \cos \varphi ds d\varphi$, where $c$ is the normalizing constant. Then the correlation for two functions $f, g \in L^2(\mu)$ is defined as:

$$C_{f,g}(n) = \int f \cdot (g \circ T^n) d\mu - \int f d\mu \int g d\mu.$$

A measure-preserving system is said to have exponential decay of correlations if there is a constant $a > 0$ such that $|C_{f,g}(n)| \leq c_{f,g} e^{-an}$ for all $n \geq 1$ and for any Holder observables $f, g$ on $M$, where $c_{f,g} > 0$ may depends on $f$ and $g$. Now it is well known that uniform hyperbolic systems and dispersing billiard systems have exponential decay of correlations. The situation may be rather complicated for convex billiard systems, since the system may only have polynomial decay of correlations. That is, there is a constant $a > 0$ such that $|C_{f,g}(n)| \leq c_{f,g} n^{-a}$ for all $n \geq 1$ and for any Holder observables $f, g$ on $M$. In fact, slow decay of correlations$^{9,10,20}$ has already been carried out for several classes of chaotic billiards including semidispersing billiards, Bunimovich-type billiards and Bunimovich stadia. We believe for the decay rates of billiard systems constructed in this paper, their general scheme should still work.

For our purposes we take the position function as the observable, that is, the projection
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of a point \(x = (s, \varphi)\) to its first coordinate \(s\). The correlations \(C_{s,s}(n)\) give us an idea of the relationship between the initial position \(s_0\) and the position \(s_n\) after \(n\)-th iterations under the billiard map \(T\). To support our observation that \(Q(1, 1)\) is an ergodic (even mixing) table, we further computed the decay rate of the position function: \(\lim_{n \to \infty} \log(C_{s,s}(n))/\log(n)\) (see Fig. 6). We can see that the limit \(\log(C_{s,s}(n))/\log(n)\) converge to \(-0.28\) as \(n \to \infty\). Therefore, the correlation function \(C_{s,s}(n)\) decays at the rate of \(\frac{1}{n^{0.28}}\). In fact, this power-law decay of correlation is quite common if the system admits many parabolic periodic orbits and hence suffers the stickiness effect caused by these orbits. We will have a detailed discussion below.

Beside the existence of the period-2 orbit described in Proposition II.1, there are four segments in the phase space of this table which are fixed by the fourth iterate \(T^4\) (see Fig. 4). They are: \(\{(s, 0) : s \text{ lies on } C_1\}\), \(\{(s, 0) : s \text{ lies on } C_2\}\), \(\{(p_1, \varphi) : \varphi \in (-\pi/3, \pi/3)\}\) and \(\{(p_2, \varphi) : \varphi \in (-\pi/3, \pi/3)\}\). Then a simple calculation shows the following (see also [18 and 21]).

**Proposition III.1.** Let \(Q(B, B)\) be the table with \(B > 1\). Every periodic orbit \(O_4\) in above families is parabolic.

**Proof.** Let us start with a periodic orbit \(O_4\) given by \((p_2, \varphi) \to (s_\varphi, 0) \to (p_2, -\varphi) \to (-s_\varphi, 0) \to (p_2, \varphi)\). The rest orbits in these families can be treated similarly. Note that the travel time \(\tau\) between each collision of this orbit is exactly 1. By (1), the tangent map of
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$T^4$ at this orbit is given by

$$D_{(p_2,\varphi)}T^4 = \left(\frac{1}{\cos \varphi} \begin{pmatrix} -1 + \cos \varphi & 1 \\ -\cos \varphi & 0 \end{pmatrix} \begin{pmatrix} 0 & 1 \\ -\cos \varphi & -1 + \cos \varphi \end{pmatrix} \right)^2 = \begin{pmatrix} -1 & 2 - 2 \cos \varphi \\ 0 & -1 \end{pmatrix}^2 = \begin{pmatrix} 1 & 4 \cos \varphi - 4 \\ 0 & 1 \end{pmatrix}. $$

So $\text{Tr}(D_{(p_2,\varphi)}T^4) = 2$ and such a periodic orbit is parabolic.

One topic of great interest in the study of billiard systems is the existence of parabolic periodic orbits and their stickiness effects on the billiard dynamics. Recall that ergodicity requires that the asymptotic distribution of a typical orbit segment converges to the smooth invariant measure. But the speed of the convergence could be slow down significantly, if the orbit runs close to some sticky periodic orbits, since the trajectory will become trapped by these parabolic periodic orbits for a long time. It can be easily seen from Fig. 5 that the trajectory approaches very slowly to theses periodic points, because once it comes close, it has to stay close to $O_4$ for a long time. This kind of orbits also exist in the annulus billiards, which have a significant effects on the dynamics. More precisely, on any annulus billiard table $A(R, B) = D_R \setminus D_1$ with $0 < B < R - 1$, there exist infinite families of parabolic periodic orbits whose trajectories avoid the inner circle but intrude into the influence disk $D_{B+1}$ of the inner disk. These are the so called Marginally Unstable Periodic Orbits (MUPO for short, see [1] for more details), which have a major effect on the ergodic properties and decay of correlations via sticking nearby orbits on the annulus table for a long time. Also see [2] for a detailed discussion of the stickiness effect of the sticky periodic orbits in stadium-type billiard systems. So the periodic orbits $O_4$ on the table $Q(B, B)$ serve as MUPOs of our billiard dynamics and should be responsible for the slow decay of the correlations of the billiard dynamics.

Next we will examine three different types of perturbations $Q(R, B)$ of the above billiard table $Q(1, 1)$, whose parameters satisfy $1 < B < R$ (corresponding to Region I in Fig. 2), as well as the line segments $1 < B = R$ and $1 = B < R$. Based on our observations, the billiard dynamics are ergodic on ‘most’ tables in these three cases. These results are particularly interesting because they provide us with some new ergodic convex billiard tables which fail the defocusing mechanism.
A. **Tables with parameters on the boundary** \(1 < B = R\)

In this subsection we investigate tables with parameters on the line segment \(1 < B = R\), which is one of the boundary of Region I in Fig. 2. We increase the parameter \(R = B\) of the table \(Q(B, B)\) from 1 to \(\infty\). As \(B\) goes to \(\infty\), the limiting table \(Q(\infty, \infty)\) will have the shape of a semidisk, on which the billiard dynamic is equivalent to the round disk table and hence is completely integrable. Moreover, due to Proposition II.1, the periodic orbit \(O_2\) is parabolic on each table in this family. However, according to our numerical results, for any finite values of \(B > 1\), these billiard systems appear to be ergodic, as we can see in Fig. 7 (with \(B = 1.0101\)) and Fig. 8 (with \(B = 100\), for an initial point \((1.5, 0.1)\), after 100,000, 1,000,000 and 100,000,000 iterations, respectively.

![Fig. 7: Trajectory segments on the phase space of the table with \(R = B = 1.0101\).](image)

A special aspect of this family of tables is that, for any value of \(B\), the center of the unit disk is always located on the boundary of \(Q(B, B)\). This leads to the existence of a family of periodic orbits of period 4. In fact, each trajectory emanating from the center of the smaller circle and hitting the boundary of \(D_1\) will lead to one of the periodic orbits, say \(O_4\). Clearly the travel time \(\tau\) between each collision equals to the radius \(r = 1\). A similar calculation as in Propostion II.1 shows that the trace is \(\text{Tr}(D_{(p_2, \phi)}T^4) = 2\). Hence all periodic orbits \(O_4\) in this family are parabolic. As already mentioned, these orbits will cause a significant slowing down effect for the convergence of time averages and the decay rate of correlations.
Besides these parabolic periodic orbits, there exist another family of nonperiodic, but sticky orbits on these tables: the sliding trajectories, which will become dominant when the corners approximate an right angle. Recall that a trajectory is sliding if it collides almost tangentially at a circular arc for many consecutive occasions. As we mentioned in the beginning of this subsection, the table $Q(B, B)$ approaches a semi-disk when $B \to \infty$. So these sticky orbits will occur when the boundary of the table created by the larger circle becomes flat enough to sustain the sliding after the trajectory bounces off of that boundary. That is, after traveling a long time almost tangentially along the unit circle, the point finally reaches the larger circle, and bounces off of that boundary while keeping almost tangential to the unit circle, since the two pieces of the table are almost perpendicular to each other. So this trajectory will stay in one side of the phase space for a tremendous iterates and only visit two small spots on the other side of the phase space once in a while. This sliding phenomenon also contributes to the significantly slowing effect on the properties of billiard dynamics on $Q(B, B)$.

B. Tables with parameters on the boundary: $1 = B < R$

In this subsection we investigate the ergodic property of billiards with parameters line on another boundary component $1 = B < R$ of Region I. We fix the distance $B = 1$ and let the
radius $R$ of the larger circle vary for all admissible values $1 < R < 2$. This corresponds to common boundary of Region I and III in the parameter space, see Fig. 2. Numerically, we let $B = 1$ in the following simulations, and gradually increase $R$ from 1.01 to 1.2. Then we pick an initial point $(1.5, 0.1)$, and run 50,000, 300,000 and 10,000,000 iterations, respectively. As one can see in Fig. 9 and 10, the phase spaces of these tables are eventually filled by iterates along one trajectory, which gives us the hint that these tables should be ergodic. However, we also notice a short interval in $1.25 < R < 1.31$, in which the billiard systems admit an elliptic island, see Section III D for further explanations.

FIG. 9: Trajectory segments on the phase space of the table $B = 1$, $R = 1.01$.

A family of period-4 orbits similar to those in Fig. 4 survive on these billiard tables. It is easy to see that all these periodic orbits are also parabolic, and will have the stickiness effect on the dynamics. Moreover, a new family of parabolic periodic orbits become dominant when $R$ gets close to $B + 1 = 2$. Recall that the periodic orbits are dense on the round disk table, and all of them are parabolic. As $R$ increases, the table $Q(R, B)$ approaches the round disk table $D_1$ and many parabolic orbits of $D_1$ survive on the table $Q(R, B)$ (see Fig. 11). These orbits correspond exactly to those MUPOs in [1] and make the convergence even slower.
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FIG. 10: Trajectory segments on the phase space of the table $B = 1$, $R = 1.11$.

FIG. 11: More and more parabolic periodic orbits appears when the table $Q(R, B)$ approaches the round disk table.

C. Tables with parameter in the interior of Region I

In this subsection we investigate ergodic property of systems with parameters in region I. Note that from Fig. 2, for tables in this region the center of $D_R$ lies outside the table $Q(R, B)$. For simplicity, we fix $R > 1$, and let $B$ vary in the range $1 < B < R$. We will denote such a table by $Q(B)$ to indicate the dependence of the table on the parameter $B$, see Fig. 12.

To demonstrate the properties of the dynamics on such tables, let us examine the cases with $R = 1.5$ fixed, and let $B$ vary from 1.01 to 1.49. We observe that, for most billiard tables in this process (except a short interval $[1.2875,1.3025]$ of $B$’s), the whole phase space is filled in by the trajectory of a single point. Moreover the distributions of the iterations are indistinguishable for several different choices of initial points, as long as the number of total
iterations is larger than 10,000. This implies that the dynamics on all three billiard tables should be ergodic. Fig. 13 shows the case with $B = 1.125$ (after 1,000,000 and 10,000,000 iterations, respectively). If we change the value of $B$ to $B = 1,1.25, 1.375$, the phase spaces of the tables $Q(B)$ behaves similarly with $Q(1.125)$.

D. The exceptions on Region I

As mentioned in Section III B and Section III C, we did identify a small set of parameters $(R, B)$, among which the billiard dynamics on the tables $Q(R, B)$ exhibit elliptic islands while satisfying $R > B \geq 1$ surrounding some elliptic periodic orbit (see Fig. 14). More precisely, denote by $\mathcal{O}_3 = \{x_i = (s_i, \varphi_i): i = 0, 1, 2\}$ the periodic-3 trajectory on $Q(R, B)$ with $x_0$ and $x_1$ sitting on the same arc. One can calculate the trace of the derivative $DT^3$.
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along this orbit, which is given by

$$\text{Tr}(D_{x_0}T^3) = 2 + \frac{8}{dd_1} (\tau_1 - d - d_1)(\tau_1 - d/2)$$

$$= -2 + \frac{8}{dd_1} (\tau_1 - d)(\tau_1 - d_1 - d/2),$$

(2)

where $d = \cos \varphi_0$, $d_1 = R \cos \varphi_2$, $\tau_1 = \tau(x_1)$.

**Proposition III.2.** Let $O_3$ be the periodic orbit given in Fig. 14. Then the orbit $O_3$ is elliptic if and only if

$$\tau_1 > d_1 + d/2.$$  

(3)

**Proof.** Note that $\tau(x) < d + d_1$ always holds on our table. Moreover it is easy to see $\tau_1 > d$ for this period-3 orbit (by drawing a perpendicular line from the center to $\tau_1$). Combining terms, we see that $\text{Tr}(D_{x_1}T^3) < 2$ by the first equality in Eq. (2). Thus the orbit $O_3$ is elliptic if and only if $\text{Tr}(D_{x_1}T^3) > -2$, which is equivalent to $\tau_1 - d_1 - d/2 > 0$ by the second equality in Eq. (2).

![FIG. 14: Period-3 orbit in the configuration space of the table $Q(R, B)$. It is elliptic if and only if $\tau_1 < d_1 + d/2$.](image)

Note that the condition (3) fails on the majority of tables $Q(R, B)$ with $1 \leq B \leq R$, since $d_1 \sim R$ can be large. Therefore the periodic orbit $O_3$ may not cause much problem for the ergodicity of most of billiard tables in Section III B and Section III C. But there do exists a tiny region on which such a periodic orbit $O_3$ exists and (3) holds along this orbit, which may fail the ergodicity of the billiard dynamics by Proposition 2. For example, if $R = 1.27$, then $\tau_1 > d_1 + d/2$ holds for all $B \in [1, 1.0198]$.

A generic feature about elliptic periodic orbit is that it is surrounded by infinitely many elliptic and hyperbolic periodic orbits of higher periods. See Fig. 15, where we show two periodic orbits of higher periods near $O_3$. 

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Doubling bifurcation: Period=6

Further bifurcation: Period=24

FIG. 15: Some periodic orbits of higher periods on billiard table with $B = 1$ and $R = 1.27$.

Moreover, we do observe a small elliptic island around the elliptic orbit $O_3$ in the phase space of the billiard table, see Fig. 16. Elliptic periodic orbits persist after small perturbations, so are the surrounding invariant curves with Diophantine rotation numbers. Therefore all tables in the nearby region of $(R, B) = (1.27, 1)$ should not be ergodic.

Remark III.3. We mainly focus on the periodic orbits of lower periods to get quantitative results. There might exist elliptic periodic orbits of higher periods. On the one hand, it is difficult to observe these orbits since the surrounding elliptic islands (if exists) might be too small and even invisible. On the other hand, periodic orbits of higher periods are sensitive to the initial conditions, may go through bifurcations (even cease to exist) after a very small changes.

FIG. 16: The phase space of $B = 1$ and $R = 1.27$, with elliptic islands surrounding the periodic orbit.

Similar phenomena are also observed on the tables with $(R, B) = (1.28, 1.02), (1.3, 1.03), (1.32, 1.06), (1.36, 1.12) (1.39, 1.16), (1.40, 1.17) (1.41, 1.19), (1.43, 1.21), (1.46, 1.25) \text{ and } (1.50, 1.29.5)$. It is interesting to note that all these tables are enclosed by two minor arcs. See Fig. 3 for the region of parameters in which the corresponding billiard systems admit
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this elliptic periodic orbit. A common feature of these tables is that the two components of their boundaries are minor arcs. All the tables $Q(R, B)$ with one major-arc are ergodic by our simulation. This also provides an motivation for our ongoing project stated in the introduction.

IV. REGION II AND III: NON-ERGODICITY AND PHASE TRANSITIONS

We have seen in the previous section that the dynamics on ‘most’ billiard tables $Q(R, B)$ in Region I are ergodic. In this section we will see that, the dynamics on every billiard table in Region II and Region III fails to be ergodic definitely. This completes the picture of different behaviors for the generalized lemon tables in Fig. 3.

First, it has been observed in the lemon billiard case\(^{18,21}\) (that is, $R = 1$), there exists an elliptic island surrounding the periodic orbit $O_2$ for all $B \neq 1$. As we will see in the following subsections, this island undergoes significant developments as the parameter $B$ moves away from 1.

A. Tables on the left boundary of Region II

We first let $B = R = 1$, and then increase $B$ gradually. From Fig. 17 and 18 we can see that, for $1 < B \leq 1.35$, the phase space of the lemon billiard $Q(B)$ consists of exactly one chaotic component and one elliptic island centered at $O_2$ (see Section II). Moreover, the size of the island grows larger and larger as we keep increasing the distance $B$. New elliptic islands start to formulate when $B$ goes over than 1.35. These new islands are centered around two periodic orbits or period six, see Fig. 19.

Orbits of the type appeared in Fig. 19 will persist for many larger values of $B$, but the shapes of the corresponding islands undergo some interesting transformations. One such example is visualized in Fig. 20, 21, 22, in which we provide three phase spaces with $B = 1.482, 1.485$ and 1.487. In these figures, new islands are created inside the the island centered at the periodic points with period 6 when we increase $B$. Moreover, as $B$ continues growing, these new-formed islands get separated from the main island and form several isolated islands. This phenomenon corresponds to a period-tripling effect, that is, these new
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islands are related to orbits of period 18. These periodic orbits are highly unstable and no longer exist when the parameter $B$ reaches 1.5.

By increasing $B$ from $B = 1.5$ to $B = 1.6$, a similar pattern, the birth and separation of new islands, is observed in the phase space of the billiard table $Q(B)$. New small islands first appear within the main center island. And they move outward as $B$ grows larger, eventually get separated from the main center island. See Fig. 23, 24 and 25.

As the parameter $B$ continues to increase, the top and bottom paths of these orbits approach the corners of the table, eventually leading to their geometric destruction and the disappearance of the related islands in phase space. Finally, as $B$ approaches 2 the ergodic portions of the phase space shrink significantly (see Fig. 26). We get a degenerate table when $B = 2$ and an empty table when $B > 2$ since the two circles no longer intersect.

These islands correspond to period orbits. For example, a pair of periodic six orbits illustrated in Fig. 27 and 28.

B. Tables on the left boundary of Region III

Now let’s move to the tables $Q(R, B)$ with $R = 1$, $B$ varies from 1 to 0. We can see the bifurcation phenomena from a completely chaotic table ($B = 1$) to a completely integrable
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As in the large distance case, the period two orbit of the type seen in Fig. 4 creates islands in phase space which will persist for all choices of $B$ we consider here (see Fig. 29 and 30).

Moreover, for all tables with $R = 1$ and $B < \sqrt{2}$, there are two special periodic orbits (the square orbits of period 4 in Fig. 37), one given by

$$(s_1, \frac{\pi}{4}) \rightarrow (-s_1, \frac{\pi}{4}) \rightarrow (s_2, \frac{\pi}{4}) \rightarrow (-s_2, \frac{\pi}{4}) \rightarrow (s_1, \frac{\pi}{4}),$$

and another one given by reversing the direction of the trajectory. These periodic orbits

![Fig. 21: $B = 1.485$](image1)

![Fig. 22: $B = 1.487$](image2)

![Fig. 23: $B = 1.54$](image3)

![Fig. 24: $B = 1.56$](image4)

![Fig. 25: $B = 1.58$](image5)

![Fig. 26: $B = 1.999$](image6)
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**FIG. 27:** The orbits corresponding to the outlying islands in Fig. 19 with $R = 1, B = 1.37$.

**FIG. 28:** Period 6 orbit corresponding to Fig. 25 with $R = 1, B = 1.58$.

**FIG. 29:** $B = 0.99$

**FIG. 30:** $B = 0.75$

disappear as $B$ goes above the critical value $\sqrt{2}$ due to the geometric destruction.

**Proposition IV.1.** This 4-period orbit of the billiard map on the table $Q(1, B)$ is hyperbolic if $B > 1/\sqrt{2}$, is parabolic if $B = 1/\sqrt{2}$ and is elliptic if $B < 1/\sqrt{2}$.

**Proof.** Firstly we note that the travel time from $(s_1, \pi/4)$ to $(−s_1, \pi/4)$ is $\tau_0 = \sqrt{2}$, and the time from $(−s_1, \pi/4)$ to $(s_2, \pi/4)$ is $\tau_1 = \sqrt{2} − B$. As usual we compute the derivative matrix of $T^4$ at this periodic orbit and find the trace formula

$$\text{Tr}(D_{(s_1, \pi/4)}T^4) = \left(2 - \frac{4\sqrt{2}B}{1}\right)^2 - 2. \quad (4)$$

In particular $\text{Tr}(D_{(s_1, \pi/4)}T^4) > 2$ if $B > \frac{1}{\sqrt{2}}$, and the hyperbolicity of this periodic orbit follows. The other two conclusions follow from (4) similarly.

We can see from Proposition IV.1 that the hyperbolicity of these orbits get weaker and weaker as we decrease $B$ (while keeping $R = 1$). Then these orbits lose their hyperbolicity and become parabolic orbits exactly at $B = 1/\sqrt{2}$. Finally they turn into (and stay as) elliptic orbits after $d$ passes this critical value (until $B$ reaches 0, at when the billiard table
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is a round table and all orbits are parabolic). We can also see this transformation from Fig. 31 to Fig. 34, that the periodic orbit $O_4$ gets separated from the chaotic sea and develops an elliptic island around it (the four thin islands surrounding the main island in Fig. 33, whom develop to thick islands in Fig. 34).

![FIG. 31: $B = 0.73$](image1)
![FIG. 32: $B = 0.72$](image2)
![FIG. 33: $B = 0.7$](image3)
![FIG. 34: $B = 0.69$](image4)

As $B$ decreases, the current islands keep developing and some new elliptic periodic orbits and the corresponding elliptic islands emerge. See Fig. 35, for $B = 0.5$, where we can observe the new island surrounding the periodic orbit given by Fig. 38; and Fig. 36 for $B = 0.3$, where new islands emerges surrounding the periodic orbits given by Fig. 39. Similar catalogue of periodic orbits have also been observed on lemon-shaped billiards with parabolic boundary arcs$^{16}$ and with elliptical hyperbolic boundaries arcs$^{17}$.

![FIG. 35: $B = 0.5$](image5)
![FIG. 36: $B = 0.3$](image6)
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FIG. 37: Square orbit for $R = 1, B < \sqrt{2}$.

FIG. 38: Triangular periodic orbits for $R = 1, B = 0.5$.

FIG. 39: The hexagonal and period 8 orbits on the table with $R = 1, B = 0.3$.

The table approaches the circular table as we continue decreasing $B$. This trend is clear from Fig. 40 and 41, where more islands appear in this process. In the phase space of the billiard table with $B = 0.01$, the islands are getting more “flattened”, and approach horizontal lines as $B$ shrinks. Finally for $B = 0$, each island has been completely flattened, that is, the phase space is foliated by horizontal lines. From the investigation of this class of billiards, it is clear that the periodic orbits in these tables and their corresponding islands in phase space play a crucial role in the transition from an ergodic billiard table to a table on which the dynamics is completely integrable.

FIG. 40: $B = 0.1$.

FIG. 41: $B = 0.01$. 

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C. Tables in the interior of II and III

We observe a similar dynamical behavior on the tables in the interior of Region II ($1 < R < B$) and Region III ($B < 1 < R$). Fig. 42 shows the trajectory segments of first 100,000,000 iterations on phase spaces for on the tables $Q(R,B)$ with $(R,B) = (1.0101, 1.0202), (1.1111, 1.2222)$ and $(1.4286, 1.8571)$, respectively. Fig. 43 shows the trajectory segments of first 100,000,000 iterations on phase spaces for on the tables $Q(R,B)$ with $(R,B) = (1.1111, 0.8888), (1.1111, 0.6667)$ and $(1.25, .05)$, respectively. These figures resemble those in [18, Fig. 4] and [21, Fig. 4], just lose the symmetry of the distributions of the trajectory segments on the phase space when $R = r = 1$.

FIG. 42: Trajectory segments on the phase space of the table $Q(R,B)$ with parameters
$(R,B) = (1.1111, 1.2222)$ and $(1.4286, 1.8571)$, respectively.

FIG. 43: Trajectory segments on the phase space of the table $Q(R,B)$ with parameters
$(R,B) = (1.1111, 0.6667)$ and $(1.25,.05)$, respectively.
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