Practice problems from old exams for math 233

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Disclaimer: Your instructor covers far more materials that we can possibly fit into a four/five questions exams. These practice tests are meant to give you an idea of the kind and varieties of questions that were asked within the time limit of that particular tests. In addition, the scope, length and format of these old exams might change from year to year. Users beware! These are NOT templates upon which future exams are based, so don't expect your exam to contain problems exactly like the ones presented here. Check the course web page for an update on the material to be covered on each exam or ask your instructor.

1 Practice problems for Exam 1.

Fall 2008 Exam

1. (a) Find parametric equations for the line $\mathbf{L}$ which contains $A(1,2,3)$ and $B(4,6,5)$.

   (b) Find parametric equations for the line $\mathbf{L}$ of intersection of the planes $x-2y+z=10$ and $2x+y-z=0$.

2. (a) Find an equation of the plane which contains the points $P(-1,0,1)$, $Q(1,-2,1)$ and $R(2,0,-1)$.

   (b) Find the distance $\mathbf{D}$ from the point $(1,6,-1)$ to the plane $2x+y-2z=19$.

   (c) Find the point Q in the plane $2x+y-2z=19$ which is closest to the point $(1,6,-1)$. (Hint: You can use part b) of this problem to help find Q or first find the equation of the line $\mathbf{L}$ passing through Q and the point $(1,6,-1)$ and then solve for Q.)

3. (a) Find the volume $\mathbf{V}$ of the parallelepiped such that the following four points $A=(3,4,0)$, $B=(3,1,-2)$, $C=(4,5,-3)$, $D=(1,0,-1)$ are vertices and the vertices B,C,D are all adjacent to the vertex A.

   (b) Find the center and radius of the sphere $x^2-4x+y^2+4y+z^2=8$.

4. (a) The position vector of a particle moving in space equals $\mathbf{r}(t)=\ t^2\mathbf{i}-t^2\mathbf{j}+\frac{1}{2}t^2\mathbf{k}$ at any time $t\geq 0$.
   Find an equation of the tangent line to the curve at the point $(4,-4,2)$.

   (b) Find the length $\mathbf{L}$ of the arc traveled from time $t=1$ to time $t=4$.

   (c) Suppose a particle moving in space has velocity

   $$\mathbf{v}(t)=\langle \sin t, 2\cos 2t, 3e^t\rangle$$

   and initial position $\mathbf{r}(0)=\langle 1,2,0\rangle$. Find the position vector function $\mathbf{r}(t)$.

5. (a) Consider the points $A(2,1,0)$, $B(3,0,2)$ and $C(0,2,1)$. Find the area of the triangle ABC. (Hint: If you know how to find the area of a parallelogram spanned by 2 vectors, then you should be able to solve this problem.

   (b) Three of the four vertices of a parallelogram are $P(0,-1,1)$, $Q(0,1,0)$ and $R(2,1,1)$. Two of the sides are PQ and PR. Find the coordinates of the fourth vertex.

Spring 2008 Exam

6. (a) Find an equation of the plane through the points $A = (1, 2, 3)$, $B = (0, 1, 3)$, and $C = (2, 1, 4)$.

   (b) Find the area of the triangle with vertices at points A, B, and C given above. *Hint: the area of this triangle is related to the area of a certain parallelogram*

7. (a) Find the parametric equations of the line passing through the point $(2, 4, 1)$ that is perpendicular to the plane $3x - y + 5z = 77$.

   (b) Find the intersection point of the line in part (a) and the plane $3x-y+5z = 77$.

8. (a) A *plane* curve is given by the graph of the vector function

$$\mathbf{u}(t) = \langle 1 + \cos t, \sin t\rangle, \quad 0 \le t \le 2\pi.$$

   Find a single equation for the curve in terms of x and y, by eliminating t.

   (b) Consider the *space* curve given by the graph of the vector function

$$\mathbf{r}(t) = \langle 1 + \cos t, \sin t, t\rangle, \quad 0 \le t \le 2\pi.$$

   Sketch the curve and indicate the direction of increasing t in your graph.

   (c) Determine parametric equations for the line T tangent to the graph of the *space* curve for $\mathbf{r}(t)$ at $t = \pi/3$, and sketch T in the graph obtained in part **(b)**.

9. Suppose that $\mathbf{r}(t)$ has derivative $\mathbf{r}'(t) = \langle -\sin 2t, \cos 2t, 0\rangle$ on the interval $0 \le t \le 1$. Suppose we know that $\mathbf{r}(0) = \langle \frac{1}{2}, 0, 1\rangle$.

   (a) Determine $\mathbf{r}(t)$ for all t.

   (b) Show that $\mathbf{r}(t)$ is orthogonal to $\mathbf{r}'(t)$ for all t.

   (c) Find the arclength of the graph of the vector function $\mathbf{r}(t)$ on the interval $0 \le t \le 1$.

10. If $\mathbf{r}(t) = (2t)\mathbf{i} + (t^2 - 6)\mathbf{j} - (\frac{1}{3}t^3)\mathbf{k}$ represents the position vector of a moving object (where $t \ge 0$ is measured in seconds and distance is measured in feet),

   (a) Find the speed and the velocity of the object at time t.

   (b) If a second object travels along a path given defined by the graph of the vector function $\mathbf{w}(s) = \langle 2, 5, 1\rangle + s\langle 2, -1, -5\rangle$, show that the paths of the two objects intersect at a common point P.

   (c) If $s = t$ in part **(b)**, (i.e. the position of the second object is $\mathbf{w}(t)$ when the first object is at position $\mathbf{r}(t)$), do the two objects ever collide?

Spring 2007 Exam

11. (a) Find parametric equations for the line which contains $A(7, 6, 4)$ and $B(4, 6, 5)$.

   (b) Find the parametric equations for the line of intersection of the planes $x - 2y + z = 5$ and $2x + y - z = 0$.

12. (a) Find an equation of the plane which contains the points $P(-1,0,2)$, $Q(1,-2,1)$ and $R(2,0,-1)$.

    (b) Find the distance from the point $(1,0,-1)$ to the plane $2x+y-2z=1$.

    (c) Find the point P in the plane $2x+y-2z=1$ which is closest to the point $(1,0,-1)$. (Hint: You can use part b) of this problem to help find P or first find the equation of the line passing through P and the point $(1,0,-1)$ and then solve for P.)

13. (a) Consider the two space curves

$$\mathbf{r}_1(t)=\langle\cos(t-1),t^2-1,2t^4\rangle,\quad \mathbf{r}_2(s)=\langle 1+\ln s,s^2-2s+1,2s^2\rangle,$$

    where t and s are two independent real parameters. Find the cosine of the angle between the tangent vectors of the two curves at the intersection point $(1,0,2)$.

    (b) Find the center and radius of the sphere $x^2+y^2+2y+z^2+4z=20$.

14. The velocity vector of a particle moving in space equals $\mathbf{v}(t)=2t\mathbf{i}-2t\mathbf{j}+t\mathbf{k}$ at any time $t\geq 0$.

    (a) At the time $t=4$, this particle is at the point $(0,5,4)$. Find an equation of the tangent line to the curve at the time $t=4$.

    (b) Find the length of the arc traveled from time $t=2$ to time $t=4$.

    (c) Find a vector function which represents the curve of intersection of the cylinder $x^2+y^2=1$ and the plane $x+2y+z=4$.

15. (a) Consider the points $A(2,1,0)$, $B(1,0,2)$ and $C(0,2,1)$. Find the area of the triangle ABC. (Hint: If you know how to find the area of a parallelogram spanned by 2 vectors, then you should be able to solve this problem.)

    (b) Suppose a particle moving in space has velocity

$$\mathbf{v}(t)=\langle\sin t,\cos 2t,e^t\rangle$$

    and initial position $\mathbf{r}(0)=\langle 1,2,0\rangle$. Find the position vector function $\mathbf{r}(t)$.

Fall 2007 Exam

16. Find the equation of the plane containing the lines

$$x=4-4t,\quad y=3-t,\quad z=1+5t\quad\text{and}$$
$$x=4-t,\quad y=3+2t,\quad z=1$$

17. Find the distance between the point $P(3,-2,7)$ and the plane given by

$$4x-6y-z=5.$$

18. Determine whether the lines L_1 and L_2 given below are parallel, skew or intersecting. If they intersect, find the point of intersection.

$$L_1 : \frac{x}{1} = \frac{y-1}{2} = \frac{z-2}{3}$$
$$L_2 : \frac{x-3}{-4} = \frac{y-2}{-3} = \frac{z-1}{2}$$

19. (a) Suppose a particle moving in space has the velocity

$$\mathbf{v}(t) = \langle 3t^2, 2\sin(2t), e^t \rangle.$$

Find the acceleration of the particle. Write down a formula for the speed of the particle (you do not need to simplify the expression algebraically).

(b) If initially the particle has the position $\mathbf{r}(0) = \langle 0, -1, 2 \rangle$, what is the position at time t?

20. Three of the four vertices of a parallelogram are $P(0, -1, 1)$, $Q(0, 1, 0)$ and $R(3, 1, 1)$. Two of the sides are PQ and PR. This problem continues on the next page.

(a) Find the area of the parallelogram.

(b) Find the cosine of the angle between the vector $\overrightarrow{PQ}$ and $\overrightarrow{PR}$.

(c) Find the coordinates of the fourth vertex.

21. Let C be the parametric curve

$$x = 2 - t^2, y = 2t - 1, z = \ln t.$$

This problem continues on the next page.

(a) Determine the point(s) of intersection of C with the xz-plane.

(b) Determine the parametric equation of the tangent line to C at $(1, 1, 0)$.

(c) Set up, but not solve, a formula that will determine the length of C for $1 \leq t \leq 2$.

Fall 2006 Exam

22. (a) Find parametric equations for the line which contains $A(2, 0, 1)$ and $B(-1, 1, -1)$.

(b) Determine whether the lines $l_1 : x = 1 + 2t,\ y = 3t, z = 2 - t$ and $l_2 : x = -1 + s,\ y = 4 + s,\ z = 1 + 3s$ are parallel, skew or intersecting.

23. (a) Find an equation of the plane which contains the points $P(-1, 2, 1)$, $Q(1, -2, 1)$ and $R(1, 1, -1)$.

(b) Find the distance from the point $(1, 2, -1)$ to the plane $2x + y - 2z = 1$.

24. (a) Let two space curves

$$\mathbf{r}_1(t) = \langle \cos(t-1), t^2 - 1, t^4 \rangle, \quad \mathbf{r}_2(s) = \langle 1 + \ln s, s^2 - 2s + 1, s^2 \rangle,$$

be given where t and s are two independent real parameters. Find the cosine of the angle between the tangent vectors of the two curves at the intersection point $(1, 0, 1)$.

(b) Suppose a particle moving in space has velocity

$$\mathbf{v}(t) = \langle \sin t, \cos 2t, e^t \rangle$$

and initial position $\mathbf{r}(0) = \langle 1, 2, 0 \rangle$. Find the position vector function $\mathbf{r}(t)$.

25. (a) Let $f(x, y) = e^{x^2 - y} + x\sqrt{4 - y^2}$. Find partial derivatives f_x, f_y and f_{xy}.

(b) Find an equation for the tangent plane of the graph of

$$f(x, y) = \sin(2x + y) + 1$$

at the point $(0, 0, 1)$.

26. (a) Let $g(x, y) = ye^x$. Estimate $g(0.1, 1.9)$ using the linear approximation of $g(x, y)$ at $(x, y) = (0, 2)$.

(b) Find the center and radius of the sphere $x^2 + y^2 + z^2 + 6z = 16$.

(c) Let $f(x, y) = \sqrt{16 - x^2 - y^2}$. Draw a contour map of level curves $f(x, y) = k$ with $k = 1, 2, 3$. Label the level curves by the corresponding values of k.

These problems are from older exams

27. Consider the line L through points $A = (2, 1, -1)$ and $B = (5, 3, -2)$. Find the intersection of the line L and the plane given by $2x - 3y + 4z = 13$.

28. Two masses travel through space along space curve described by the two vector functions

$$\mathbf{r}_1(t) = \langle t, 1 - t, 3 + t^2 \rangle, \ \mathbf{r}_2(s) = \langle 3 - s, s - 2, s^2 \rangle$$

where t and s are two independent real parameters.

(a) Show that the two space curves intersect by finding the point of intersection and the parameter values where this occurs.

(b) Find parametric equation for the tangent line to the space curve $\mathbf{r}(t)$ at the intersection point.

29. Consider the parallelogram with vertices A, B, C, D such that B and C are adjacent to A. If $A = (2, 5, 1)$, $B = (3, 1, 4)$, $D = (5, 2, -3)$, find the point C.

30. Consider the points $A = (2, 1, 0)$, $B = (1, 0, 2)$ and $C = (0, 2, 1)$.

(a) Find the orthogonal projection $proj_{\overrightarrow{AB}}(\overrightarrow{AC})$ of the vector $\overrightarrow{AC}$ onto the vector $\overrightarrow{AB}$.

(b) Find the area of triangle ABC.

(c) Find the distance d from the point C to the line L that contains points A and B.

31. Find parametric equations for the line of intersection of the planes $x - 2y + z = 1$ and $2x + y + z = 1$.

32. Let L_1 denote the line through the points $(1,0,1)$ and $(-1,4,1)$ and let L_2 denote the line through the points $(2,3,-1)$ and $(4,4,-3)$. Do the lines L_1 and L_2 intersect? If not, are they skew or parallel?

33. (a) Find the volume of the parallelepiped such that the following four points $A = (1,4,2)$, $B = (3,1,-2)$, $C = (4,3,-3)$, $D = (1,0,-1)$ are vertices and the vertices B, C, D are all adjacent to the vertex A.

    (b) Find an equation of the plane through A, B, D.

    (c) Find the angle between the plane through A, B, C and the xy plane.

34. The velocity vector of a particle moving in space equals $\mathbf{v}(t) = 2t\mathbf{i} + 2t^{1/2}\mathbf{j} + \mathbf{k}$ at any time $t \geq 0$.

    (a) At the time $t = 0$ this particle is at the point $(-1,5,4)$. Find the position vector $\mathbf{r}(t)$ of the particle at the time $t = 4$.

    (b) Find an equation of the tangent line to the curve at the time $t = 4$.

    (c) Does the particle ever pass through the point $P = (80,41,13)$?

    (d) Find the length of the arc traveled from time $t = 1$ to time $t = 2$.

35. Consider the surface $x^2 + 3y^2 - 2z^2 = 1$.

    (a) What are the traces in $x = k, y = k, z = k$? Sketch a few.

    (b) Sketch the surface in the space.

36. Find an equation for the tangent plane to the graph of $f(x,y) = y \ln x$ at $(1,4,0)$.

37. Find the distance between the given parallel planes

$$z = 2x + y - 1, \quad -4x - 2y + 2z = 3.$$

38. Identify the surface given by the equation $4x^2 + 4y^2 - 8y - z^2 = 0$. Draw the traces and sketch the curve.

39. A projectile is fired from a point 5 m above the ground at an angle of 30 degrees and an initial speed of 100 m/s.

    a) Write an equation for the acceleration vector.
    b) Write a vector for initial velocity.
    c) Write a vector for initial position.
    d) At what time does the projectile hit the ground?
    e) How far did it travel, horizontally, before it hit the ground?

40. Explain why the limit of $f(x,y) = (3x^2y^2)/(2x^4 + y^4)$ does not exist as (x,y) approaches $(0,0)$.

41. Find an equation of the plane that passes through the point $P(1,1,0)$ and contains the line given by parametric equations $x = 2 + 3t,\ y = 1 - t,\ z = 2 + 2t$.

42. Find all of the first order and second order partial derivatives of the function.

    (a) $f(x,y) = x^3 - xy^2 + y$

(b) $f(x,y) = \ln(x + \sqrt{x^2 + y^2})$

43. Find the linear approximation of the function $f(x,y) = xye^x$ at $(x,y) = (1,1)$, and use it to estimate $f(1.1, 0.9)$.

44. Find a vector function which represents the curve of intersection of the paraboloid $z = 2x^2 + y^2$ and the parabolic cylinder $y = x^2$.

2 Practice problems for Exam 2.

Fall 2008 Exam

1. (a) For the function $f(x,y) = 2x^2 + xy^2$, calculate f_x, f_y, f_{xy}, f_{xx}:
   - $f_x(x,y) =$
   - $f_y(x,y) =$
   - $f_{xy}(x,y) =$
   - $f_{xx}(x,y) =$

   (b) What is the **gradient** $\nabla f(x,y)$ of f at the point $(1,2)$? $\nabla f =$

   (c) Calculate the **directional derivative** of f at the point $(1,2)$ in the direction of the vector $\mathbf{v} = \langle 3,4 \rangle$?

   (d) Next evaluate $D_{\mathbf{u}}f(1,2) =$

   (e) What is the **linearization** $L(x,y)$ of f at $(1,2)$?

   (f) Use the **linearization** $L(x,y)$ in the previous part to estimate $f(0.9, 2.1)$.

2. A hiker is walking on a mountain path. The surface of the mountain is modeled by $z = 100 - 4x^2 - 5y^2$. The positive x-axis points to **East** direction and the positive y-axis points **North**.

   (a) Suppose the hiker is now at the point $P(2,-1,79)$ heading North, is she **ascending** or **descending**?

   (b) When the hiker is at the point $Q(1,0,96)$, in which direction on the map should she initially head to **descend** most rapidly?

   (c) What is her **rate** of **descent** when she travels at a speed of 10 meters per minute in the direction of maximal decent from $Q(1,0,96)$?

   (d) When the hiker is at the point $Q(1,0,96)$, in which two directions on her map can she initially head to **neither** ascend nor descend (to keep traveling at the same height)?
   Justify your answers.

3. (a) Let $f(x,y)$ be a differentiable function with the following values of the **partial derivatives** $f_x(x,y)$ and $f_y(x,y)$ at certain points (x,y)

x	y	$f_x(x,y)$	$f_y(x,y)$
1	1	-2	4
-1	2	3	-1
1	2	-1	3

(You are given more values than you will need for this problem.) Suppose that x and y are functions of variable t: $x = t^3$; $y = t^2 + 1$, so that we may regard f as a function of t. Compute the derivative of f with respect to t when $t = 1$. Use the **Chain Rule** to find $\frac{\partial z}{\partial v}$ when $u = 1$ and $v = 1$, where

$$z = x^3y^2 + y^3x; \quad x = u^2 + v^2,\ y = u - v^2.$$

(b) Use the **Chain Rule** to find $\frac{\partial z}{\partial v}$ when $u = 1$ and $v = 1$, where

$$z = x^3y^2 + y^3x; \quad x = u^2 + v^2,\ y = u - v^2.$$

4. Consider the surface $x^2 + y^2 - 2z^2 = 0$ and the point $P(1, 1, 1)$ which lies on the surface.

   (i) Find the equation of the **tangent plane** to the surface at P.

   (ii) Find the equation of the **normal line** to the surface at P.

5. Let

$$f(x, y) = 2x^3 + xy^2 + 6x^2 + y^2.$$

   Find and classify (as local **maxima**, local **minima** or **saddle** points) all **critical points** of f.

6. A flat circular plate has the shape of the region $x^2 + y^2 \leq 4$. The plate (including the boundary $x^2 + y^2 = 4$) is heated so that the temperature at any point (x, y) on the plate is given by $T(x, y) = x^2 + y^2 - 2x$. Find the temperatures at the **hottest** and the coldest points on the plate, including the boundary $x^2 + y^2 = 4$.

Spring 2008 Exam

7. Consider the equation $x^2 + y^2/9 + z^2/4 = 1$.

   (a) Identify this quadric (i.e. quadratic surface), and graph the portion of the surface in the region $x \geq 0, y \geq 0$, and $z \geq 0$. Your graph should include tick marks along the three positive coordinate axes, and must clearly show where the surface intersects any of the three positive coordinate axes.

   (b) Calculate z_x and z_y at an arbitrary point (x, y, z) on the surface.

   (c) Determine the equation of the tangent plane to the surface at the point $(\frac{1}{\sqrt{2}}, \frac{3}{2}, 1)$.

8. Given the function $f(x, y) = x^2y + ye^{xy}$.

   (a) Find the linearization of f at the point $(0, 5)$ and use it to approximate the value of f at the point $(.1, 4.9)$. (An unsupported numerical approximation to $f(.1, 4.9)$ will not receive credit.)

   (b) Suppose that $x(r, \theta) = r\cos\theta$ and $y(r, \theta) = r\sin\theta$. Calculate f_θ at $r = 5$ and $\theta = \frac{\pi}{2}$.

(c) Suppose a particle travels along a path $(x(t), y(t))$, and that $F(t) = f(x(t), y(t))$ where $f(x, y)$ is the function defined above. Calculate $F'(3)$, assuming that at time $t = 3$ the particle's position is $(x(3), y(3)) = (0, 5)$ and its velocity is $(x'(3), y'(3)) = (3, -2)$.

9. Consider the function $f(x, y) = 2\sqrt{x^2 + 4y}$.

 (a) Find the directional derivative of $f(x, y)$ at $P = (-2, 3)$ in the direction starting from P pointing towards $Q = (0, 4)$.

 (b) Find all unit vectors $\mathbf{u}$ for which the directional derivative $D_{\mathbf{u}}f(-2, 3) = 0$.

 (c) Is there a unit vector $\mathbf{u}$ for which the directional derivative $D_{\mathbf{u}}f(-2, 3) = 4$? Either find the appropriate $\mathbf{u}$ or explain why not.

10. let $f(x, y) = \frac{2}{3}x^3 + \frac{1}{3}y^3 - xy$.

 (a) Find all critical points of $f(x, y)$.

 (b) Classify each critical point as a relative maximum, relative minimum or saddle; you do not need to calculate the function at these points, but your answer must be justified by an appropriate calculation.

11. Use the method of Lagrange multipliers to determine all points (x, y) where the function $f(x, y) = 2x^2 + 4y^2 + 16$ has an extreme value (either a maximum or a minimum) subject to the constraint $\frac{x^2}{4} + y^2 = 4$.

Fall 2007 Exam

12. Find the x and y coordinates of all critical points of the function

$$f(x, y) = 2x^3 - 6x^2 + xy^2 + y^2$$

and use the second derivative test to classify them as local minima, local maxima or saddle points.

13. A hiker is walking on a mountain path. The surface of the mountain is modeled by $z = 1 - 4x^2 - 3y^2$. The positive x-axis points to East direction and the positive y-axis points North. Justify your answers.

 (a) Suppose the hiker is now at the point $P(\frac{1}{4}, -\frac{1}{2}, 0)$ heading North, is she ascending or descending?

 (b) When the hiker is at the point $Q(\frac{1}{4}, 0, \frac{3}{4})$, in which direction should she initially head to ascend most rapidly?

14. Find the volume of the solid bounded by the surface $z = 6 - xy$ and the planes $x = 2$, $x = -2$, $y = 0$, $y = 3$, and $z = 0$.

15. Let $z(x,y) = x^2 + y^2 - xy$ where $x = s - r$ is a known function of r and s and $y = y(r,s)$ is an unknown function of r and s. (Note that z can be considered a function of r and s.) Suppose we know that

$$y(2,3) = 3, \quad \frac{\partial y}{\partial r}(2,3) = 7, \quad \text{and} \quad \frac{\partial y}{\partial s}(2,3) = -5.$$

Calculate $\frac{\partial z}{\partial r}$ when $r = 2$ and $s = 3$.

16. Let $F(x,y,z) = x^2 - 2xy - y^2 + 8x + 4y - z$. This problem continues on the next page.

    (a) Write the equation of the tangent plane to the surface given by $F(x,y,z) = 0$ at the point $(-2,1,-5)$.

    (b) Find the point (a,b,c) on the surface at which the tangent plane is horizontal, that is, parallel to the $z = 0$ plane.

17. Find the points on the ellipse $x^2 + 4y^2 = 4$ that are closest to the point $(1,0)$.

Fall 2006 Exam

18. (a) Let $f(x,y)$ be a differentiable function with the following values of the partial derivatives $f_x(x,y)$ and $f_y(x,y)$ at certain points (x,y):

x	y	$f_x(x,y)$	$f_y(x,y)$
1	1	-2	4
-1	2	3	-1
1	2	-1	1

(You are given more values than you will need for this problems.) Suppose that x and y are functions of variable t:

$$x = t^3; \quad y = t^2 + 1,$$

so that we may regard f as a function of t. Compute the derivative of f with respect to t when $t = 1$.

    (b) Use the Chain Rule to find $\frac{\partial z}{\partial v}$ when $u = 1$ and $v = 1$, where

$$z = x^3y^2 + y^3x; \quad x = u^2 + v, \quad y = 2u - v.$$

19. (a) Let $f(x,y) = x^2y^3 + y^4$. Find the directional derivative of f at the point $(1,1)$ in the direction which forms an angle (counterclockwise) of $\pi/6$ with the positive x-axis.

    (b) Find an equation of the tangent line to the curve $x^2y + y^3 - 5 = 0$ at the point $(x,y) = (2,1)$.

20. Let

$$f(x,y) = 2x^3 + xy^2 + 5x^2 + y^2.$$

Find and classify (as local maxima, local minima or saddle points) all critical points of f.

21. Find the maximum value of $f(x,y) = 2x^2 + y^2$ on the circle $x^2 + y^2 = 1$.

22. Find the volume above the rectangle $-1 \le x \le 1, \quad 2 \le y \le 5$ and below the surface $z = 5 + x^2 + y$.

23. Evaluate the integral

$$\int_0^1 \int_{\sqrt{y}}^1 \sqrt{x^3+1}\, dx\, dy$$

by reversing the order of integration.

These problems from older exams

24. Use Chain Rule to find dz/dt or $\partial z/\partial u$, $\partial z/\partial v$.

(1) $z = x^2y + 2y^3, \quad x = 1 + t^2, \quad y = (1-t)^2.$

(2) $z = x^3 + xy^2 + y^3, \quad x = uv, \quad y = u + v.$

25. If $z = f(x,y)$, where f is differentiable, and $x = 1 + t^2,\ y = 3t$, compute dz/dt at $t = 2$ provided that $f_x(5,6) = f_y(5,6) = -1$.

26. For the following functions
(1). $f(x,y) = x^2y + y^3 - y^2$, (2) $g(x,y) = x/y + xy$, (3) $h(x,y) = \sin(x^2y) + xy^2$.

(a) Find the gradient.
(b) Find the directional derivative at the point $(0,1)$ in the direction of $\mathbf{v} = \langle 3,4\rangle$.
(c) Find the maximum rate of change at the point $(0,1)$.

27. Find an equation of the tangent plane to the surface $x^2 + 2y^2 - z^2 = 5$ at the point $(2,1,1)$.

28. Find parametric equations for the tangent line to the curve of intersection of the surfaces $z^2 = x^2 + y^2$ and $x^2 + 2y^2 + z^2 = 66$ at the point $(3,4,5)$.

29. Find and classify all critical points (as local maxima, local minima, or saddle points) of the following functions.
(1) $f(x,y) = x^2y^2 + x^2 - 2y^3 + 3y^2$, (2) $g(x,y) = x^3 + y^2 + 2xy - 4x - 3y + 5$.

30. Find the minimum value of $f(x,y) = 3 + xy - x - 2y$ on the closed triangular region with vertices $(0,0)$, $(2,0)$ and $(0,3)$.

31. Use Lagrange multipliers to find the extreme values of the following functions with the given constraint.
(1) $f(x,y) = xy$ with constraint $x^2 + 2y^2 = 3$;
(2) $g(x,y,z) = x + 3y - 2z$ with constraint $x^2 + 2y^2 + z^2 = 5$.

32. Find the following iterated integrals.
(1) $\int_1^4 \int_0^2 (x + \sqrt{y})\, dx\, dy$

(2) $\int_1^2 \int_0^1 (2x + 3y)^2\, dy\, dx$

(3) $\int_0^1 \int_x^{2-x} (x^2 - y)\, dy\, dx$

(4) $\int_0^1 \int_{x^2}^1 x^3 \sin(y^3)\, dy\, dx$ (hint: reverse the order of integration)

33. Evaluate the following double integrals.
(1) $\int\int_R \cos(x + 2y)\, dA,\ R = \{(x, y) \mid 0 \leq x \leq \pi,\ 0 \leq y \leq \pi/2\}$

(2) $\int\int_R e^{y^2}\, dA,\ R = \{(x, y) \mid 0 \leq y \leq 1,\ 0 \leq x \leq y\}$

(3) $\int\int_R x\sqrt{y^2 - x^2}\, dA,\ R = \{(x, y) \mid 0 \leq y \leq 1,\ 0 \leq x \leq y\}$

34. Find the volume.
(1) The solid under the surface $z = 4 + x^2 - y^2$ and above the rectangle

$$R = \{(x, y) \mid -1 \leq x \leq 1,\ 0 \leq y \leq 2\}$$

(2) The solid under the surface $z = 2x + y^2$ and above the region bounded by curves $x - y^2 = 0$ and $x - y^3 = 0$.

3 Practice problems for Final Exam.

Fall 2008 Exam

1. (a) Consider the points
$A = (1, 0, 0),\ \ B = (2, 1, 0)$ and $C = (1, 2, 3)$. Find the **parametric equations** for the line **L** passing through the points A and C.
(b) Find an equation of the plane in $\mathbb{R}^3$ which contains the points $A,\ B,\ C$.
(c) Find the area of the triangle **T** with vertices A, B and C.

2. Find the volume under the graph of $f(x, y) = x + 2xy$ and over the bounded region in the first quadrant $\{(x, y) \mid x \geq 0,\ \ y \geq 0\}$ bounded by the curve $y = -x^2 + 1$ and the x and y-axes.

3. Let
$$\mathbf{I} = \int_0^1 \int_{2x}^2 \sin(y^2)\, dy\, dx.$$
(a) Sketch the region of integration.

(b) Write the integral **I** with the order of integration reversed.

(c) Evaluate the integral **I**. Show your work.

4. Consider the function $\mathbf{F}(x, y, z) = x^2 + xy^2 + z$.

   (a) What is the gradient $\nabla\mathbf{F}(x, y, z)$ of **F** at the point $(1, 2, -1)$?

   (b) Calculate the directional derivative of **F** at the point $(1, 2, -1)$ in the direction of the vector $\langle 1, 1, 1\rangle$?

   (c) What is the maximal rate of change of **F** at the point $(1, 2, -1)$?

   (d) Find the equation of the tangent plane to the level surface $\mathbf{F}(x, y, z) = 4$ at the point $(1, 2, -1)$.

5. Find the volume **V** of the solid under the surface $z = 1 - x^2 - y^2$ and above the xy-plane.

6. (a) Determine whether the following vector fields are **conservative** or not. Find a **potential function** for those which are indeed **conservative**.

   i. $\mathbf{F}(x, y) = \langle x^2 + e^x + xy, xy - \sin(y)\rangle$.

   ii. $\mathbf{G}(x, y) = \langle 3x^2y + e^x + y^2, x^3 + 2xy + 3y^2\rangle$.

7. Evaluate the line integral

$$\int_C yz\ dx + xz\ dy + xy\ dz,$$

where C is the curve starting at $(0, 0, 0)$, traveling along a line segment to $(1, 0, 0)$ and then traveling along a second line segment to $(1, 2, 1)$.

8. (a) Use Green's Theorem to show that if $\mathbf{D} \subset \mathbb{R}^2$ is the bounded region with boundary a positively oriented simple closed curve C, then the area of **D** can be calculated by the formula:

$$\textbf{Area} = \frac{1}{2}\int_C -y\ dx + x\ dy$$

   (b) Consider the ellipse $4x^2 + y^2 = 1$. Use the above area formula to calculate the area of the region $\mathbf{D} \subset \mathbb{R}^2$ with boundary this ellipse. (Hint: This ellipse can be parametrized by $\mathbf{r}(t) = \langle \frac{1}{2}\cos(t), \sin(t)\rangle$ for $0 \le t \le 2\pi$.)

Spring 2008 Exam

9. Use the space curve $\vec{r}(t) = \langle t^2 - 1, t^2, t/2\rangle$ for parts (a) and (b)below.

   (a) Find the velocity, speed, and acceleration of a particle whose positions function is $\vec{r}(t)$ at time $t = 4$.

   (b) Find all points where the particle with position vector $\vec{r}(t)$ intersects the plane $x + y - 2z = 0$.

10. Let D be the region of the xy plane above the graph of $y = x^2$ and below the line $y = x$.

(a) Determine an iterated integral expression for the double integral

$$\int_D xy\,dA$$

(b) Find an equivalent iterated integral to the one found in part (a) with the reversed order of integration.

(c) Evaluate one of the two iterated integrals in parts (a,b).

11. Find the absolute maximum and absolute minimum values of $f(x, y) = x^2 + 2y^2 - 2y$ in the set $D = \{(x, y) : x^2 + y^2 \leq 4\}$.

12. Let D be the region in the first quadrant $x, y \geq 0$ that lies between the two circles $x^2 + y^2 = 4$ and $x^2 + y^2 = 9$.

    (a) Describe the region D using polar coordinates.

    (b) Evaluate the double integral

$$\int\int_D 3x + 3y\;dA.$$

13. (a) Find $\frac{\partial z}{\partial x}$ and $\frac{\partial z}{\partial y}$ at the point$(1, 0, -1)$ for the implicit function z determined by the equation

$$x^3 + y^3 + z^3 - 3xyz = 0.$$

    (b) Is the tangent plane to the surface

$$x^3 + y^3 + z^3 - 3xyz = 0$$

    at the point $(1, 0, -1)$ perpendicular to the plane $2x + y - 3z = 2$? Justify your answer with an appropriate calculation.

14. (a) Consider the vector field $\mathbf{G}(x, y) = \langle 4x^3 + 2xy, x^2\rangle$. Show that $\mathbf{G}$ is *conservative* (i.e. $\mathbf{G}$ is a *potential* or a *gradient* vector field), and use the fundamental theorem for line integrals to determine the value of

$$\int_C \mathbf{G}\cdot d\mathbf{r},$$

    where C is the contour consisting of the line starting at (2,-2) and ending at $(-1, 1)$.

    (b) Now let T denote the closed contour consisting of the triangle with vertices at (0,0),(1,0), and (1,1) with the counterclockwise orientation, and let $\mathbf{F}(x, y) = \langle \frac{1}{2}y^2 - y, xy\rangle$. Compute

$$\int_C \mathbf{F}\cdot d\mathbf{r}$$

    directly (from the definition of line integral).

    (

    (c) Explain how Green's theorem can be used to show that the integral in **(b)** must be equal to the area of the region D interior to T.

Fall 2007 Exam

15. Let
$$I = \int_0^4 \int_{\sqrt{y}}^2 e^{x^3}\,dx\,dy.$$
   (a) Sketch the region of integration
   (b) Write the integral I with the order of integration reversed.
   (c) Evaluate the integral I. Show your work.

16. Find the distance from the point $(3, 2, -7)$ to the line
$$x = 1 + t, \quad y = 2 - t, \quad z = 1 + 3t.$$

17. (a) Find the velocity and acceleration of a particle moving along the curve
$$\mathbf{r}(t) = \langle t, t^2, t^3 \rangle$$
   at the point $(2, 4, 8)$.
   (b) Find all points where the curve in part (a) intersects the surface $z = 3x^3 + xy - x$.

18. Find the volume of the solid which lies below the sphere $x^2 + y^2 + z^2 = 4$, above the xy-plane, and inside the cylinder $x^2 + y^2 = 3$.

19. Consider the line integral
$$\int_C \sqrt{1+x}\,dx + 2xy\,dy,$$
where C is the triangular path starting from $(0, 0)$, to $(2, 0)$, to $(2, 3)$, and back to $(0, 0)$. This problem continues on the next page.
   (a) Evaluate this line integral directly, **without** using Green's Theorem.
   (b) Evaluate this line integral using Green's theorem.

20. Consider the vector field $\mathbf{F} = (y^2/x^2)\mathbf{i} - (2y/x)\mathbf{j}$.
   (a) Find a function f such that $\nabla f = \mathbf{F}$.
   (b) Let C be the curve given by $\mathbf{r}(t) = \langle t^3, \sin t \rangle$ for $\frac{\pi}{2} \le t \le \pi$. Evaluate the line integral $\int_C \mathbf{F} \cdot d\mathbf{r}$.

Fall 2006 Exam

21. Find parametric equations for the line in which the planes $x - 2y + z = 1$ and $2x + y + z = 1$ intersect.

22. Consider the surface $x^2 + y^2 - 2z^2 = 0$ and the point $P(1, 1, 1)$ which lies on the surface.
   (i) Find the equation of the tangent plane to the surface at P.
   (ii) Find the equation of the normal line to the surface at P.

23. Find the maximum and minimum values of the function

$$f(x, y) = x^2 + y^2 - 2x$$

on the disc $x^2 + y^2 \leq 4$.

24. Evaluate the iterated integral

$$\int_0^1 \int_0^{\sqrt{1-x^2}} \sqrt{x^2 + y^2}\, dy\, dx.$$

25. Find the volume of the solid under the surface $z = 4 - x^2 - y^2$ and above the region in the xy plane between the circles $x^2 + y^2 = 1$ and $x^2 + y^2 = 4$.

26. Determine whether the following vector fields are conservative or not. Find a potential function for those which are indeed conservative.

    (a) $\mathbf{F}(x, y) = (x^2 + xy)\mathbf{i} + (xy - y^2)\mathbf{j}$.

    (b) $\mathbf{F}(x, y) = (3x^2 y + y^2)\mathbf{i} + (x^3 + 2xy + 3y^2)\mathbf{j}$.

27. Evaluate the line integral

$$\int_C (x^2 + y)\, dx + (xy + 1)\, dy$$

where C is the curve starting at $(0, 0)$, traveling along a line segment to $(1, 2)$ and then traveling along a second line segment to $(0, 3)$.

28. Use Green's Theorem to evaluate the line integral

$$\int_C \mathbf{F} \cdot d\mathbf{r}$$

where $\mathbf{F} = \langle y^3 + \sin 2x, 2xy^2 + \cos y \rangle$ and C is the unit circle $x^2 + y^2 = 1$ which is oriented counterclockwise.

These problems are from older exams

29. (a) Express the double integral

$$\int\int_R x^2 y - x\, dA$$

as an iterated integral and evaluate it, where R is the first quadrant region enclosed by the curves $y = 0$, $y = x^2$ and $y = 2 - x$.

    (b) Find an equivalent iterated integral expression for the double integral in 29a, where the order of integration is reserved from the order used in part 29a. (Do **not** evaluate this integral.)

30. Calculate the line integral

$$\int_C \mathbf{F} \cdot \mathbf{dr},$$

where $\mathbf{F}(x, y) = y^2 x\mathbf{i} + xy\mathbf{j}$, and C is the path starting at $(1, 2)$, moving along a line segment to $(3, 0)$ and then moving along a second line segment to $(0, 1)$.

31. Evaluate the integral
$$\int\int_R y\sqrt{x^2+y^2}\,dA$$
with R the region $\{(x,y)\colon 1 \le x^2+y^2 \le 2,\ 0 \le y \le x.\}$

32. (a) Show that the vector field $\mathbf{F}(x,y) = \left\langle \frac{1}{y} + 2x, -\frac{x}{y^2} + 1 \right\rangle$ is conservative by finding a potential function $f(x,y)$.
    (b) Let C be the path described by the parametric curve $\mathbf{r}(t) = \langle 1+2t, 1+t^2\rangle$ for 32a to determine the value of the line integral $\int_C \mathbf{F}\cdot d\mathbf{r}$.

33. (a) Find the equation of the tangent plane at the point $P = (1, 1-1)$ in the level surface $f(x,y,z) = 3x^2 + xyz + z^2 = 1$.
    (b) Find the directional derivative of the function $f(x,y,z)$ at $P = (1,1,-1)$ in the direction of the tangent vector to the space curve $\mathbf{r}(t) = \langle 2t^2 - t, t^{-2}, t^2 - 2t^3\rangle$ at $t = 1$.

34. Find the absolute maxima and minima of the function
$$f(x,y) = x^2 - 2xy + 2y^2 - 2y.$$
in the region bounded by the lines $x = 0$, $y = 0$ and $x + y = 7$.

35. Consider the function $f(x,y) = xe^{xy}$. Let P be the point $(1,0)$.
    (a) Find the rate of change of the function f at the point P in the direction of the point $(3,2)$.
    (b) Give a direction in terms of a unit vector (there are two possibilities) for which the rate of change of f at P in that direction is zero.

36. (a) Find the work done be the vector field $\mathbf{F}(x,y) = \langle x-y, x\rangle$ over the circle $\mathbf{r}(t) = \langle \cos t, \sin t\rangle$, $0 \le t \le 2\pi$.
    (b) Use Green's Theorem to calculate the line integral $\int_C (-y^2)\,dx + xy\,dy$, over the **positively** (counterclockwise) oriented closed curve defined by $x = 1,\ y = 1$ and the coordinate axes.

37. (a) Show that the vector field $\mathbf{F}(x,y) = \langle x^2 y, \frac{1}{3}x^3\rangle$ is conservative and find a function f such that $\mathbf{F} = \nabla f$.
    (b) Using the result in part **a**, calculate the line integral $\int_C \mathbf{F}\cdot d\mathbf{r}$, along the curve C which is the arc of $y = x^4$ from $(0,0)$ to $(2,16)$.

38. Consider the surface $x^2 + y^2 - \frac{1}{4}z^2 = 0$ and the point $P(1,2,-2\sqrt{5})$ which lies on the surface.
    (a) Find the equation of the tangent plane to the surface at the point P.
    (b) Find the equation of the normal line to the surface at the point P.

39. A flat circular plate has the shape of the region $x^2 + y^2 \le 1$. The plate (including the boundary $x^2 + y^2 = 1$) is heated so that the temperature at any point (x,y) on the plate is given by
$$T(x,y) = x^2 + 2y^2 - x$$

Find the temperatures at the hottest and the coldest points on the plate, including the boundary $x^2 + y^2 = 1$.

40. The acceleration of a particle at any time t is given by

$$\mathbf{a}(t) = \langle -3\cos t, -3\sin t, 2\rangle,$$

while its initial velocity is $v(0) = \langle 0, 3, 0\rangle$. At what times, if any are the velocity and the acceleration of the particle orthogonal?

41. Find parametric equations for the line in which the planes $3x - 6y - 2z = 15$ and $2x + y - 2z = 5$ intersect.

42. Find the equation of the plane containing the points $P(1, 3, 0)$, $Q(2, -1, 2)$ and $R(0, 0, 1)$.

43. Find all points of intersection of the parametric curve $\mathbf{r}(t) = \langle 2t^2 - 2, t, 1 - t - t^2\rangle$ and the plane $x + y + z = 3$.

44. Find the absolute maximum and minimum of the function $f(x, y) = x^2 + 2y^2 - 2y$ on the closed disc $x^2 + y^2 \le 5$ of radius $\sqrt{5}$.

45. Evaluate

$$\int\int_R xy\, dA,$$

where R is the region in the first quadrant bounded by the line $y = 2x$ and the parabola $y = x^2$.

46. Consider the vector field $\mathbf{F}(x, y) = \langle 2xy + \sin y, x^2 + x\cos y + 1\rangle$.

    (a) Show that $\mathbf{F}(x, y) = \langle 2xy + \sin y, x^2 + x\cos y + 1\rangle$ is conservative by finding a potential function $f(x, y)$ for $\mathbf{F}(x, y)$.

    (b) Use your answer to part **a** to evaluate the line integral $\int_C \mathbf{F} \cdot d\mathbf{r}$, where C is the arc of the parabola $y = x^2$ going from $(0, 0)$ to $(2, 4)$.

47. Evaluate the line integral

$$\int_C \mathbf{F} \cdot d\mathbf{r}$$

where $\mathbf{F} = \langle y^2 + \sin x, xy\rangle$ and C is the unit circle oriented counterclockwise.

48. Evaluate the line integral $\int_C \mathbf{F} \cdot d\mathbf{r}$, where $\mathbf{F} = \langle y^2, 2xy + x\rangle$ and C is the curve starting at $(0, 0)$, traveling along a line segment to $(2, 1)$ and then traveling along a second line segment to $(0, 3)$.