Some Curiosities in 4 Dimensions

Menageries of Smooth Exotica

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Intuitively: An n-dimensional topological manifold is a nice (paracompact, separable) topological space which is locally Euclidean of dimension n.

Throughout, we will take *topological n-dimensional manifold* to mean a Hausdorff, metrizable topological space M together with an atlas of charts $\{(U_\alpha, \varphi_\alpha)\}_{\alpha \in A}$, where the U_α form an open covering of M, i.e. $M = \cup_\alpha U_\alpha$, and the maps $\varphi_\alpha \colon U_\alpha \to \varphi(U_\alpha) \subseteq \mathbb{R}^n$ are homeomorphisms onto their images, which are open sets of $\mathbb{R}^n$.

The integer n is called the dimension of the manifold. We will ultimately focus on the case when $n = 4$.

An atlas is called *smooth*, or $\mathcal{C}^\infty$, if the charts for any two overlapping sets of the cover induce a *diffeomorphism* (a $\mathcal{C}^\infty$ map with $\mathcal{C}^\infty$ inverse):

$$\text{For all } \alpha, \beta \in A \text{ such that } U_{\alpha\beta} := U_\alpha \cap U_\beta \neq \varnothing,$$

$$\varphi_{\alpha\beta} := \varphi_\beta \circ \varphi_\alpha^{-1}\big|_{\varphi_\alpha(U_{\alpha\beta})} : \varphi_\alpha(U_{\alpha\beta}) \to \varphi_\beta(U_{\alpha\beta})$$

is a $\mathcal{C}^\infty$ map with $\mathcal{C}^\infty$ inverse $\varphi_{\beta\alpha}$ defined in the obvious way. We say that the charts $(U_\alpha, \varphi_\alpha)$, and (U_β, φ_β) are $\mathcal{C}^\infty$-compatible.

A smooth structure on a topological n-manifold M is a choice of a *maximal smooth atlas*, i.e. a smooth atlas $\mathcal{A} := \{(U_\alpha, \phi_\alpha)\}_{\alpha \in A}$ such that for any chart (V, ψ) smoothly compatible with $(U_\alpha, \varphi_\alpha)$ for all $\alpha \in A$ is itself in the atlas: $(V, \psi) \in \mathcal{A}$.

- If $\mathcal{A}$ is a $\mathcal{C}^\infty$ atlas on a topological manifold M, then there is a unique maximal atlas on M which contains $\mathcal{A}$, so in particular a smooth atlas on M, if one exists, determines a smooth structure on M.
- One can ask if a given topological manifold supports distinct (i.e. inequivalent) smooth structures. This was answered in the affirmative, as we'll discuss.
- Two manifolds M and N are *diffeomorphic* if there is a $\mathcal{C}^\infty$ map $F\colon M \to N$ which is a homeomorphism with $\mathcal{C}^\infty$ inverse $F^{-1}\colon N \to M$.
- Let M and N be homeomorphic spaces which are Hausdorff and metrizable and each supporting smooth structures. If there is no diffeomorphism $F\colon M \to N$, then the smooth structures they support are distinct. We call such a pair of homeomorphic but not diffeomorphic manifolds an *exotic pair*.

Notational conventions

- Throughout, X will denote an abstract, closed (compact, boundaryless and connected), oriented, topological 4-manifold.
- $\chi(X)$ will denote the Euler characteristic of X, which by Poincaré duality may be computed as $2 - 2b_1 + b_2$, where $b_1 := \mathrm{rank}\, H_1(X;\mathbb{Z})$ and $b_2 := \mathrm{rank}\, H_2(X;\mathbb{Z})$ are the first and second Betti numbers. Integral homology groups will be abbreviated to omit the coefficients, $H_k(X) := H_k(X;\mathbb{Z})$.
- Σ_g will denote a genus g orientable surface (there's a unique orientable genus g surface up to diffeomorphism),
- $\mathbb{D}^n$ will denote an n-dimensional disk, and $\mathbb{S}^n = \partial\mathbb{D}^{n+1}$ an n-dimensional sphere.

- If Y, Z are manifolds of dimension n then their connect sum, denoted $Y\#Z$, is the manifold obtained by deleting an n-disk in the interior of each, and then gluing the resulting $\mathbb{S}^{n-1}$ boundary components:

$$Y\#Z = (Y \setminus \mathbb{D}^n) \cup_{\partial\mathbb{D}^n} (Y \setminus \mathbb{D}^n)\,.$$

  Iterated connect sums are indicated by putting the number of summands after the sum sign: e.g. $\#3X := X\#X\#X$.

- Some common 4-manifolds we will see are the complex projective plane $\mathbb{CP}^2$, elliptic surfaces $E(n)$, Dolgachev surfaces $E(n)_{p,q}$, and various surface bundles over surfaces.

- In 1956, John Milnor described the first exotic pair: he constructed a seven dimensional manifold by considering certain $\mathbb{S}^3$ bundles over $\mathbb{S}^4$, with structure group $\mathrm{SO}(4)$. Using Morse theory he could show that his examples were homeomorphic to $\mathbb{S}^7$. He constructed an invariant (Milnor's λ invariant) which distinguished the smooth structures on his examples from the standard smooth structure on $\mathbb{S}^7$. This gives the first historical counterexample to the smooth generalized Poincaré conjecture.
- Stephen Smale proved the h-cobordism theorem in 1962. The power of this theorem is to simplify many arguments in the differential topology of higher dimensional manifolds.

- Two n-manifolds M and N are *h-cobordant* if there is an $n+1$ dimensional manifold W such that $\partial W = \overline{M} \amalg N$ and the inclusions of M and N into W are homotopy equivalences. The h-cobordism theorem tells us that if $n \geq 5$, and M and N are simply connected manifolds h-cobordant through a manifold W, then W is diffeomorphic to $M \times \mathbb{D}^1$, and hence M and N are diffeomorphic. The proof relies on Morse theory and a technical result involving embedded disks called "the Whitney trick".
- The method of proof fails smoothly in dimension 4. However, in 1981 Michael Freedman proved the topological h-cobordism theorem for 4-manifolds. The result hinges on finding a way to surmount the problem of self-intersecting disks in a topological handle decomposition of the the topological manifold underlying the cobordism. Thus, a 4D topological "Whitney trick" was found. The trade off: Casson handles, which are homeomorphic to $\mathbb{D}^2 \times \mathbb{R}^2$ but smoothly inequivalent to standard (open) 2-handles.

- S. K. Donaldson developed polynomial invariants, coming from Yang-Mills Theory (a non-abelian gauge theory) which detect smooth structure. He used these to give the first counterexamples to a smooth 4D version of the h-cobordism. In particular, he proved that Dolgachev surfaces $D(n)_{p,q}$ yield infinitely many exotica homeomorphic to $\mathbb{CP}^2\#9\overline{\mathbb{CP}^2}$ for $n > 1$, and p, q coprime.

- In 1982 Freedman found an example of an exotic $\mathbb{R}^4$, and by 1985 Robert Gompf had shown that there were infinitely many exotic structures on $\mathbb{R}^4$, in contrast to all other dimensions, where there are at most finitely many smooth structures on a given manifold. Indeed for $n \neq 4$, $\mathbb{R}^n$ has a unique smooth structure!

The 4D landscape

- Classifying all smooth oriented closed 4-manifolds by homotopy type is untenable: given any finitely presented group G, we can construct a smooth, closed, oriented 4-manifold X (more generally, an n-manifold for any $n \geq 4$) which has $\pi_1(X) \cong G$. Thus the classification is at least as hard as classifying finitely presented groups, which is impossible (in the sense of no algorithm can be produced), given the various unsolvable decision problems related to group presentations.

- A more feasible problem is the classification of smooth, closed, simply connected 4-manifolds, at least up to homotopy.

- The work of Freedman tells us that such X are indeed classified up to *homeomorphism* by an algebro-topological invariant called the *intersection form*.

Surfaces and Intersection Forms

- Every element of $H_2(X)$ can be represented by an embedded surface $\Sigma \hookrightarrow X$. This is true more generally for any oriented, smooth 4-manifold, possibly with boundary, and an analogous result holds for non-orientable surfaces and the second homology with $\mathbb{Z}/2\mathbb{Z}$ coefficients.

- Given a closed smooth 4-manifold X, one can define a unimodular symmetric bilinear form

$$Q_X \colon H_2(X)/\mathrm{Torsion} \times H_2(X)/\mathrm{Torsion} \to \mathbb{Z}\,,$$

via the Kronecker pairing

$$Q_X([\Sigma],[\Sigma']) = \langle PD[\Sigma] \smile PD[\Sigma'], [X]\rangle\,.$$

By assuming our representative surfaces Σ and Σ' are generically embedded so as to intersect transversely, $Q_X([\Sigma],[\Sigma'])$ is just the signed number of intersections of Σ and Σ'.

- For a simply connected closed 4-manifold X, the only interesting integer homology group is $H_2(X)$ ($H_0(X) \cong \mathbb{Z} \cong H_4(X)$, while $H_1(X) = 0 = H_3(X)$, by the universal coefficients theorem, Poincaré duality and simple connectivity). Thus one might hope and even suspect that the intersection form Q_X is a useful invariant.

- Unimodular symmetric bilinear integral forms come in two flavors: definite and indefinite. The indefinite ones are completely classified by their numerical invariants: rank, signature, and type (also called parity).

Freedman's theorem (1982) tells us that for every unimodular symmetric bilinear integral form Q there exists a simply connected closed topological 4-manifold X such that $Q \cong Q_X$. Built upon the work of Whitehead and Milnor showing that intersection forms characterize homotopy type, Freedman proved that intersection forms are a complete invariant of homeomorphism type of smooth, closed, simply connected 4-manifolds.

Donaldson's theorem states that if Q_X is the intersection form of a simply connected closed smooth 4-manifold, and if Q_X is definite, then it must be diagonalizable over the integers. Thus, invoking Freedman and the existence of unimodular symmetric bilinear integral forms Q_X which are definite and non-diagonalizable, we know there exist non-smoothable 4-manifolds. Henceforth, our focus will be on closed *smooth* 4-manifolds X which are simply connected.

Donaldson's theorem departs from the purely algebro-topological machinery we've seen so far. In particular the proof requires *differential geometry* (and in particular, a version of Yang-Mills theory)! Assume Q_X is negative definite. Roughly, one endows X with a Riemannian metric, and studies connections and curvature forms on some $\mathbb{C}^2$-bundle $E \to X$ of Euler number 1. A connection ∇ has associated to it a curvature 2-form F_∇. The Hodge star operator on 4D 2-forms gives us a decomposition into self-dual and anti-self dual parts. We can then seek connections whose curvature form has vanishing anti-self dual component.

One then studies the moduli space, i.e. the space of all solutions modulo *gauge equivalence* (meaning solutions up to equivalence under the action of automorphisms of the bundle E). The remarkable fact is that after a slight perturbation and removal of a finite collection of singular points, this space is an open 5-manifold! Donaldson discovered that this space can be naturally compactified with X as a boundary, and after deleting neighborhoods of the singularities, one obtains an h-cobordism to $\#r\overline{\mathbb{CP}^2}$, where $r = \mathrm{rank}\, Q_X$.

Donaldson developed invariants from the study of Yang-Mills equations and the moduli space. In 1987 he provided the first example of a closed simply connected exotic 4-manifold via an $H^2(X;\mathbb{Z})$ valued invariant Γ_X built using the topology of the Moduli space. The examples he applied this invariant to were Dolgachev surfaces, which are elliptically fibered complex surfaces obtained from simpler elliptic surfaces using a process called logarithmic transformation (which is topologically a torus surgery, where the torus is a fiber.)

Many of the Gauge theory results of Donaldson can be re-proven with an abelian gauge theory called Seiberg-Witten theory. Here, one studies the moduli space of solutions to certain elliptic PDEs generalizing the Dirac equation (beloved grandparent to quantum field theory). One obtains invariants which are used particularly effectively in the study of exotic symplectic 4-manifolds.

Another powerful tool put to use in the quest to detect and understand exotica are various Floer homologies (a kind of infinite dimensional version of Morse homology).

Infinite Exotic Families

Using tools from symplectic and complex geometry, there's been a veritable race to produce various infinite families of exotic closed simply connected 4-manifolds, with an emphasis on finding examples with small Euler characteristic. The original examples, Dolgachev surfaces, are elliptic fibrations, with the "smallest" examples being $E(1)_{p,q}$, having $\chi(E(1)_{p,q}) = 12$. There are plenty of recent examples of strides towards ever smaller Euler characteristic. For example, recently R. Inanç Baykur and Mustafa Korkmaz have exhibited exotic $\mathbb{CP}^2\#4\overline{\mathbb{CP}^2}$. There are many examples of families of exotic $\#m\mathbb{CP}^2\#n\overline{\mathbb{CP}^2}$ for various $m + n > 5$.

Exotic Knottings of surfaces

Given two (not necessarily orientable) embedded surfaces $\Sigma \hookrightarrow X$, $\Sigma' \hookrightarrow X$, we say that they are an exotic pair of embeddings if there is a homeomorphism of pairs $(X, \Sigma) \cong (X, \Sigma')$ but no diffeomorphism of these pairs. If $X = \mathbb{S}^4$, then we can define a standard "unknotted" embedding (up to ambient isotopy) of any closed surface Σ. If (X, Σ) is homeomorphic to the standard pair (X, Σ_{std}) for the unknotted embedding, but not diffeomorphic to it, we say Σ is an exotic knotting of Σ_{std}.

In 1987, S. Finashin, O. Viro, and M. Kreck proved the existence of an infinite family of exotic knottings of $\#10\mathbb{RP}^2$ in $\mathbb{S}^4$, a phenomenon unique to 4-dimensions.

They used equivariant rational tangle surgery on Dolgachev surfaces with a complex involution, and showed that their construction realized the Dolgachev surfaces as a branched cover over $\mathbb{S}^4$ with branch locus the "real locus" of fixed points of the complex involution. The embeddings are mutually topologically equivalent, but if the resulting embeddings were smoothly equivalent, then the diffeomorphisms of pairs would lift to diffeomorphisms of Dolgachev surfaces, contradicting their known exoticness. Using other exotica, one can construct exotic embeddings of any $\#m\mathbb{RP}^2$ in $\mathbb{S}^4$ for $m \geq 10$.

M. Kreck showed in 1990 that all these embeddings were unknotted. In 2007 Finashin improved upon the original result, getting exotic embeddings of $\#m\mathbb{RP}^2$ in $\mathbb{S}^4$ for $m \geq 6$.

Exotic Group Actions on 4-Manifolds

A group action on a manifold is called exotic if the group action is equivariantly homeomorphic but not equivariantly diffeomorphic. Ron Fintushel and Ron Stern first showed the existence of orientation reversing exotic actions on $\mathbb{S}^4$ in 1981. In 2009 Fintushel, Stern, and Sunukjian produced orientation *preserving* cyclic exotic actions on simply connected 4-manifolds with interesting gauge theory (namely, manifolds with non-trivial Seiberg-Witten invariants).

One can also differentiate between smoothly and symplectically or holomorphically equivariant actions. Weimin Chen and Slawomir Kwasik have investigated these distinctions for $K3$ surfaces (which are complex surfaces homeomorphic to $E(2)$).

Where do Exotica in 4D come from?

- Complex surfaces and log transforms - e.g. the Dolgachev surfaces,
- Rational blowdowns and knot surgery, rational tangle surgery,
- Symplectic geometry: Luttinger surgery and "reverse engineering", symplectic fiber sums,
- Lefschetz Fibrations: combinatorially constructive from monodromy factorizations in a surface mapping class group.

Symplectic Geography vs. Botany

- Geography: Which pairs (χ, σ) are realized as the Euler characteristic and signature of a closed complex surface or closed symplectic 4-manifold? This problem is well understood and gives rise to a symplectic analogue of the classification of complex surfaces. We have many existence results, but not necessarily a good understanding of the geometry of manifolds realizing lattice points in certain regions.
- Botany: Can we classify the diffeomorphism types all simply connected 4-manifolds realizing a given pair? This problem is hard and far from resolved for most pairs.

Small exotica

How small can the Euler characteristic of a simply connected closed exotic 4-manifold be? Are there exotic $\mathbb{CP}^2$s, $\mathbb{CP}^2\#\overline{\mathbb{CP}^2}$s, or $\mathbb{S}^2\times\mathbb{S}^2$s? This is similar to the smooth generalized 4D Poincaré conjecture, in that we know almost nothing about how to construct examples or how to attempt a proof of nonexistence of exotica.

Exotic Knottings: Are there any exotic knottings of orientable surfaces in $\mathbb{S}^4$? How small can we make the genus? For nonorientable surfaces, can we improve on Finashin's lower bound? (Exotically knotted connect sums of two Klein bottles, or of one Klein bottle, or just $\mathbb{RP}^2$?)