The Relationship between Rank and Nullity

A Fundamental Theorem for Linear Algebra

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March 28, 2018

Outline

Definitions of Column Space and Null Space

Definition

Let $\mathrm{A} \in \mathbb{R}^{m\times n}$ be a real matrix. Recall

- The *column space of* A is the subspace $\mathrm{Col}\,\mathrm{A}$ of $\mathbb{R}^m$ spanned by the columns of A: $\mathrm{Col}\,\mathrm{A} = \mathrm{Span}\,\{\mathbf{a}_1, \ldots, \mathbf{a}_n\} \subseteq \mathbb{R}^m$ where $\mathrm{A} = \begin{bmatrix} \mathbf{a}_1 & \ldots & \mathbf{a}_n \end{bmatrix}$.

  Equivalently, $\mathrm{Col}\,\mathrm{A}$ is the same as the image $T(\mathbb{R}^n) \subseteq \mathbb{R}^m$of the linear map $T(\mathbf{x}) = \mathrm{A}\mathbf{x}$.

- The *null space of* A is the subspace $\mathrm{Nul}\,\mathrm{A}$ of $\mathbb{R}^n$ consisting of all vectors $\mathbf{x}$ which are solutions to the homogeneous equation with matrix A: $\mathrm{Nul}\,\mathrm{A} := \{\mathbf{x} \in \mathbb{R}^n \,|\, \mathrm{A}\mathbf{x} = \mathbf{0}\} \subseteq \mathbb{R}^n$

  Equivalently, it is the kernel of the map $T(\mathbf{x}) = \mathrm{A}\mathbf{x}$.

Rank: The Dimension of the Column Space

Definition

The *rank of a linear map* $T : V \to W$ between finite dimensional vector spaces V and W is the dimension of the image:

$$\mathrm{rank}\, T = \dim T(V)\,.$$

Given an $m \times n$ matrix A, the *rank of* A is the dimension of the column space of A:

$$\mathrm{rank}\, \mathrm{A} = \dim \mathrm{Col}\, \mathrm{A}\,.$$

Remark

Observe that $\mathrm{rank}\, T \leq \dim V$ and $\mathrm{rank}\, T \leq \dim W$.

Remark

We know $\operatorname{rank} T \leq \dim V$ because the image subspace is spanned by the images of basis vectors, and so in particular, $T(V)$ is spanned by a set of $\dim V$ vectors, which is an upper bound on the size of a linearly independent spanning set.

That $\operatorname{rank} T \leq \dim W$ follows from the fact that $T(V)$ is a subspace of W, and so its dimension is less than or equal to the dimension of W.

Nullity: The Dimension of the Null Space

Definition

The *nullity of a linear map* $T : V \to W$ between finite dimensional vector spaces V and W is the dimension of the kernel:

$$\mathrm{nullity}\, T = \dim \ker T\,.$$

Given an $m \times n$ matrix A, the *nullity of* A is the dimension of the null space of A:

$$\mathrm{nullity}\, \mathrm{A} = \dim \mathrm{Nul}\, \mathrm{A}\,.$$

Remark

Observe that $\mathrm{nullity}\, T \leq \dim V$, but it need not be bounded by the dimension of W.

Exercise

Explain the above remark about the bound on the nullity of a linear map. What is the relationship between the dimension of the codomain and the nullity?

Finding a Basis of the Column Space

To find a basis of the Column space of A, find a row equivalent matrix B is echelon form and locate the pivot columns.

Recall that the corresponding columns of A are the pivot columns of A. As each non-pivot column corresponds to a free variable, any non-pivot columns may be realized as linear combinations of the pivot columns.

Thus the *pivot columns of* A are a maximal linearly independent subset of the column of A, and span $\mathrm{Col}\,\mathrm{A}$.

That is, the pivot columns of A are a basis of the column space of A.

Finding a Basis of the Null Space

To find a basis of the null space of A, solve the homogeneous system $A\mathbf{x} = \mathbf{0}$.

The solution vector $\mathbf{x}$ can be written as a linear combination of some vectors weighted by free variables.

Since each such vector corresponds to a unique free variable, it will have a one in a coordinate position where the other vectors have zeros.

Thus, the matrix whose columns are these vectors spanning the null space has as many pivots as the system has free variables, and so this collection is linearly independent and forms a basis of the null space.

Summarizing: Computing Rank and Nullity

A basis of $\mathrm{Col}\, A$ is given by the pivot columns of A. Be careful to use the columns of the original matrix A and not of $\mathrm{RREF}(A)$! The rank is the number of pivot columns of A, or equivalently the number of pivot positions of $\mathrm{RREF}(A)$.

A basis of $\mathrm{Nul}\, A$ is found by solving the homogenous equation and decomposing a parameterized solution vector as a linear combination of vectors weighted by free variables. The vectors in this sum form a basis of $\mathrm{Nul}\, A$. The nullity of A is thus the number of free variables for the homogeneous system, which is the same as the number of non-pivot columns of A.

Definition of the Row Space of a Matrix

Definition

Let A be an $m \times n$ matrix. The set of all linear combinations of the rows of A is called the *row space* of A.

Since each row of A is a column of A^{t}, we often regard the row space as $\mathrm{Col}\,\mathrm{A}^{\mathrm{t}} \subseteq \mathbb{R}^m$, though strictly speaking, these are isomorphic but not equal spaces, and $\mathrm{Row}\,\mathrm{A} \not\subseteq \mathbb{R}^m$, as it consists of objects distinct from column vectors. Later, we'll define the correct space within which the row space is realized as a subspace.

Row Equivalence and Row Spaces

Theorem

Suppose $\mathrm{A} \sim \mathrm{B}$. *Then* $\operatorname{Row} \mathrm{A} = \operatorname{Row} \mathrm{B}$. *Moreover, if* B *is in echelon form, then the nonzero rows of* B *form a nonzero basis of both* $\operatorname{Row} \mathrm{B}$ *and* $\operatorname{Row} \mathrm{A}$.

Proof.

Suppose B is obtained from A by row operations.

Then the rows of B are linear combinations of the rows of A, and thus any linear combination of rows of B is a linear combination of rows of A, whence $\operatorname{Row} \mathrm{B} \subseteq \operatorname{Row} \mathrm{A}$.

Since elementary row operations are invertible, we can argue symmetrically that $\operatorname{Row} \mathrm{A} \subseteq \operatorname{Row} \mathrm{B}$, whence $\operatorname{Row} \mathrm{A} = \operatorname{Row} \mathrm{B}$.

Proof (continued.)

If B is in echelon form, then the nonzero rows form a linearly independent set, as no nonzero row can be made as a linear combination of the nonzero rows below it.

Equivalently, by transposing, consider that each nonzero row has a pivot position, and gives a pivot column of B^{t}.

Thus the nonzero rows of B form a basis of $\operatorname{Row}\mathrm{B} = \operatorname{Row}\mathrm{A}$. □

An Example

Example

Find bases of the row space, column space, and null space of the matrix

$$A = \begin{bmatrix} 1 & 4 & 3 & 2 & 5 \\ 4 & 8 & 12 & 9 & 0 \\ 3 & 4 & 9 & 7 & -5 \\ 2 & 8 & 6 & 5 & 6 \end{bmatrix}.$$

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$$\mathrm{A} \sim \mathrm{B} = \begin{bmatrix} 1 & 4 & 3 & 2 & 5 \\ 0 & -8 & 0 & 1 & -20 \\ 0 & 0 & 0 & 1 & -4 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

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Since B is in echelon form, we deduce that

$$\mathrm{Row}\,\mathrm{A} = \mathrm{Span}\left\{ \begin{array}{c} \begin{bmatrix} 1 & 4 & 3 & 2 & 5 \end{bmatrix}, \\ \begin{bmatrix} 0 & -8 & 0 & 1 & -20 \end{bmatrix}, \\ \begin{bmatrix} 0 & 0 & 0 & 1 & -4 \end{bmatrix} \end{array} \right\}.$$

Example

Since B has pivots in the first, second and fourth columns, we deduce that

$$\mathrm{Col}\,A = \mathrm{Span}\left\{ \begin{bmatrix} 1 \\ 4 \\ 3 \\ 2 \end{bmatrix}, \begin{bmatrix} 4 \\ 8 \\ 4 \\ 8 \end{bmatrix}, \begin{bmatrix} 2 \\ 9 \\ 7 \\ 5 \end{bmatrix} \right\}$$

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To get a basis of the null space we need to continue to row reduce until we obtain $\mathrm{RREF}(\mathrm{A})$, so that we can solve the homogeneous equation $\mathrm{A}\mathbf{x} = \mathbf{0}$.

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Example

Thus,

$$\mathrm{RREF(A)} = \mathrm{RREF(B)} = \begin{bmatrix} 1 & 0 & 3 & 0 & 5 \\ 0 & 1 & 0 & 0 & 2 \\ 0 & 0 & 0 & 1 & -4 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

which implies that the homogenous system has solution

$$\mathbf{x} = \begin{bmatrix} -3s - 5t \\ -2t \\ s \\ 4t \\ t \end{bmatrix} = s\begin{bmatrix} -3 \\ 0 \\ 1 \\ 0 \\ 0 \end{bmatrix} + t\begin{bmatrix} 5 \\ -2 \\ 0 \\ 4 \\ 1 \end{bmatrix}.$$

Example

Thus,

$$\mathrm{Nul\,A} = \mathrm{Span}\left\{ \begin{bmatrix} -3 \\ 0 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 5 \\ -2 \\ 0 \\ 4 \\ 1 \end{bmatrix} \right\},$$

and we have that $\mathrm{rank\,A} = \dim \mathrm{Col\,A} = \dim \mathrm{Row\,A} = 3$ and $\mathrm{nullity\,A} = \dim \mathrm{Nul\,A} = 2$.

A Cautionary Warning about Row Space Basis

Observe in the above example that the first three rows of A are *not* a basis of $\mathrm{Row}\,\mathrm{A}$. In particular, they are not even linearly independent! Indeed, $R_3 = R_2 - R_1$.

Thus, it is important that you take the rows from the row equivalent echelon form B to build a basis of the row space. This is in contrast to the column space basis, which must come from the *original matrix* A, and not the echelon form B.

Exercise

Show that though the first three rows of A are not a basis of $\mathrm{Row}\,\mathrm{A}$, the first, second and fourth rows do form a basis of $\mathrm{Row}\,\mathrm{A}$. It is always possible to form a basis of the row space using rows of the original matrix–how does one determine such a basis?

Rank + Nullity = # of Columns

Theorem (The Rank Nullity Theorem)

Let $\mathrm{A} \in \mathbb{R}^{m\times n}$ *be any real* $m \times n$ *matrix. Then*

$$\dim \mathrm{Row}\,\mathrm{A} = \mathrm{rank}\,\mathrm{A} = \dim \mathrm{Col}\,\mathrm{A}$$

and

$$\boxed{n = \mathrm{rank}\,\mathrm{A} + \mathrm{nullity}\,\mathrm{A}\,.}$$

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Proof.

That $\dim \mathrm{Row}\, \mathrm{A} = \dim \mathrm{Col}\, \mathrm{A} = \mathrm{rank}\, \mathrm{A}$ follows from the fact that $\mathrm{Row}\, \mathrm{A} = \mathrm{Col}\, \mathrm{A}^t$, every row of A which contains a pivot position yields both a pivot column of A and a pivot column of A^t, and the pivot columns of A span the image of the map $T(\mathbf{x}) \mapsto \mathrm{A}\mathbf{x}$.

Proof (continued.)

The equality $\mathrm{rank}\, T + \mathrm{nullity}\, T = n$ is just a restatement of the fact that the number of pivot columns plus the number of non-pivot columns is equal to the total number of columns.

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Indeed, the non-pivot columns are in one-to one correspondence with vectors in a basis for $\mathrm{Nul\,A}$, and thus the number of non-pivot columns is just the nullity. □

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Indeed, the non-pivot columns are in one-to one correspondence with vectors in a basis for $\mathrm{Nul\,A}$, and thus the number of non-pivot columns is just the nullity. □

We could use the theory of linear coordinates to prove an analogue for general linear maps. We state the analogue, but opt to prove it directly, as the direct proof contains some insights lost in a coordinate-based proof.

Rank-Nullity for General Linear Maps

Theorem (General Rank-Nullity Theorem)

Let V be a finite dimensional $\mathbb{F}$-vector space, and let $T : V \to W$ be a linear map. Then

$$\boxed{\dim_{\mathbb{F}} V = \dim_{\mathbb{F}} T(V) + \dim_{\mathbb{F}} \ker T = \mathrm{rank}\, T + \mathrm{nullity}\, T\,.}$$

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Proof.

Since V is finite dimensional, there exists a finite basis $\mathscr{B}$ of V. Moreover, since $\ker T \subset V$ is a subspace, it is itself a finite dimensional vector space, and it thus possesses a finite basis.

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Let $n := \dim_{\mathbb{F}} V$, $k = \dim_{\mathbb{F}} \ker T$, and $r := \dim_{\mathbb{F}} T(V)$. We need to prove that $k + r = n$.

Proof.

We will prove that we can construct an ordered basis $\mathscr{B} = (\mathbf{u}_1, \ldots \mathbf{u}_k, \mathbf{v}_1, \ldots \mathbf{v}_r)$ of V such that:

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1. Take any basis $\widetilde{\mathscr{B}} = \{\mathbf{b}_1, \dots \mathbf{b}_n\}$ of V and some basis $\{\mathbf{u}_1, \dots \mathbf{u}_k\}$ of $\ker T \subset V$.

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1. Take any basis $\widetilde{\mathscr{B}} = \{\mathbf{b}_1, \dots \mathbf{b}_n\}$ of V and some basis $\{\mathbf{u}_1, \dots \mathbf{u}_k\}$ of $\ker T \subset V$.

Claim

After possibly re-indexing $\widetilde{\mathscr{B}}$, replacing $\mathbf{b}_1$ by $\mathbf{u}_1$ yields a basis $(\mathbf{u}_1, \mathbf{b}_2, \dots, \mathbf{b}_n)$ of V.

Proof (continued.)

Since $\mathbf{u}_1$ is an element of a basis of $\ker T$, it is nonzero, and so may be expressed as a nontrivial linear combination of elements of $\tilde{\mathscr{B}}$: $\mathbf{u}_1 = x_1\mathbf{b}_1 + \ldots x_n\mathbf{b}_n$. Order $\tilde{\mathscr{B}}$ so that $x_1 \neq 0$. We'll prove the claim by contradiction.

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If the claim is false, then $\mathbf{u}_1 \in \mathrm{Span}_{\mathbb{F}}\{\mathbf{b}_2, \ldots, \mathbf{b}_n\}$ so there are $n-1$ constants $y_2 \ldots y_n \in \mathbb{F}$, not all equal to 0, such that $\mathbf{u}_1 = y_2\mathbf{b}_2 + \ldots + y_n\mathbf{b}_n$.

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$$\mathbf{0} = x_1\mathbf{b}_1 + (x_2 - y_2)\mathbf{b}_2 + \ldots + (x_n - y_n)\mathbf{b}_n\,.$$

Since $\tilde{\mathscr{B}}$ is a basis, this equation can only be true if all of the coefficients are 0. But $x_1 \neq 0$, which gives a contradiction. Thus the claim is proved.

Proof (continued.)

The set of elements in $\widetilde{\mathscr{B}} - \{\mathbf{u}_1\}$ is then a basis of an $n-1$ dimensional subspace complimentary to $\mathrm{Span}_{\mathbb{F}}\{\mathbf{u}_1\}$.

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Applying the claim to this subspace, we can replace an element, such as $\mathbf{b}_2$ of $\widetilde{\mathscr{B}} - \{\mathbf{u}_1\}$ by $\mathbf{u}_2$, and so forth. Iterate the process of replacement of elements of $\widetilde{\mathscr{B}}$ by elements $\mathbf{u}_i$ of the basis of $\ker T$, until the $\mathbf{u}_i$ have been exhausted.

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Let $\mathscr{B} = (\mathbf{u}_1, \ldots, \mathbf{u}_k, \mathbf{v}_1, \ldots, \mathbf{v}_r)$ be the resulting ordered basis where $\mathbf{v}_1, \ldots \mathbf{v}_r$ are the $r = n - k$ elements of $\widetilde{\mathscr{B}}$ that remain after replacing k elements by the vectors in the basis of the kernel.

Proof (continued.)

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$$\begin{aligned} T(V) &= \mathrm{Span}\,\{T(\mathbf{u}_1), \dots, T(\mathbf{u}_k), T(\mathbf{v}_1), \dots, T(\mathbf{v}_r)\} \\ &= \mathrm{Span}\,\{\mathbf{0}, \dots, \mathbf{0}, T(\mathbf{v}_1), \dots, T(\mathbf{v}_r)\} \end{aligned}$$

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$$\begin{aligned} T(V) &= \mathrm{Span}\,\{T(\mathbf{u}_1), \dots, T(\mathbf{u}_k), T(\mathbf{v}_1), \dots, T(\mathbf{v}_r)\} \\ &= \mathrm{Span}\,\{\mathbf{0}, \dots, \mathbf{0}, T(\mathbf{v}_1), \dots, T(\mathbf{v}_r)\} \\ &= \mathrm{Span}\,\{T(\mathbf{v}_1), \dots, T(\mathbf{v}_r)\} \end{aligned}$$

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Thus the set $\{T(\mathbf{v}_1), \dots T(\mathbf{v}_r)\}$ spans the image.

We need to show that this set is linearly independent. We prove this by contradiction as well.

Proof (continued.)

Suppose that there is a nontrivial relation $\sum_{i=1}^{r} a_i T(\mathbf{v}_i) = \mathbf{0}$.

Proof (continued.)

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$$T\left(\sum_{i=1}^r a_i \mathbf{v}_i\right) = \mathbf{0} \implies \sum_{i=1}^r a_i \mathbf{v}_i \in \ker T.$$

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$$T\left(\sum_{i=1}^r a_i \mathbf{v}_i\right) = \mathbf{0} \implies \sum_{i=1}^r a_i \mathbf{v}_i \in \ker T.$$

Since $\{\mathbf{u}_1, \ldots, \mathbf{u}_k\}$ is a basis of $\ker T$, we then can express the linear combination of $\mathbf{v}_i$s as a linear combination of the $\mathbf{u}_j$s:

$$\sum_{i=1}^r a_i \mathbf{v}_i = \sum_{j=1}^k b_j \mathbf{u}_j \,.$$

Proof (continued.)

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We thus obtain a relation

$$a_1\mathbf{v}_1 + \ldots + a_r\mathbf{v}_r - b_1\mathbf{u}_1 - \ldots - b_k\mathbf{u}_k = \mathbf{0},$$

and since at least one of the a_is is nonzero, this relation is nontrivial.

Proof (continued.)

This contradicts the linear independence of the elements of $\mathscr{B}$, so, the assumption that there exists a non-trivial linear relation on the set $\{T\mathbf{v}_1, \ldots T\mathbf{v}_r\}$ is untenable.

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We conclude that $\{T\mathbf{v}_1, \ldots T\mathbf{v}_r\}$ is a basis of the image, so the rank is then r. It is therefore clear that $n = r + k$, so

$$\dim_{\mathbb{F}} V = \dim_{\mathbb{F}} T(V) + \dim_{\mathbb{F}} \ker T = \mathrm{rank}\, T + \mathrm{nullity}\, T.$$

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Challenge Problem: Use the theory of linear coordinates to give an alternate proof of the theorem. (Note: one must be careful to check that various statements made about matrix representatives of a linear map hold regardless of choices of basis.)

Interpreting Rank-Nullity: Injective Maps

Let's examine the conceptual consequences of this theorem briefly.

First, note that if a map $T: V \to W$ is an injection (one-to one map) from a finite dimensional vector space V, then the kernel has dimension 0, and by rank-nullity we have that the dimension of the image is the same as the dimension of the domain.

In particular, if a linear map is injective, its image is an *"isomorphic copy" of the domain*, and one may refer to such maps as *linear embeddings*, since we can imagine that we are identifying the domain with its image as a subspace of the target space.

Interpreting Rank-Nullity: Surjective Maps

If we have a surjective map $T: V \to W$ from a finite dimensional vector space V, then the image has the same dimension as W.

We see that the dimensions then satisfy

$$\dim \ker T = \dim V - \dim W \,,$$

whence we see that the nullity is the difference in the dimensions of the domain and codomain for a surjective map.

We can interpret this as follows: to cover the space W linearly by V, we have to squish extra dimensions, nullifying a subspace (the kernel) whose dimension is complimentary to $\dim W$.

Interpreting Rank-Nullity: Invertible Maps

Finally of course, in a linear isomorphism $T\colon V \to W$, we have injectivity and surjectivity, and so in particular we have $\mathrm{nullity}\, T = 0$ and $\dim V = \dim W = \mathrm{rank}\, T$.

Following the ideas of the proof, we see that for an isomorphism, the basis has no elements $\mathbf{u}_i$ coming from a basis of the kernel (since the kernel is trivial), and that any basis $\mathscr{B} = (\mathbf{v}_1, \ldots, \mathbf{v}_n)$ of V transforms to a basis $T(\mathscr{B}) = \big(T(\mathbf{v}_1), \ldots, T(\mathbf{v}_n)\big)$ of W. Conversely, given a $1:1$ map of a basis of V to a basis of W, we get an isomorphism by linear extension, as discussed previously.

Interpreting Rank-Nullity: the General Case

For a general map $T : V \to W$, not necessarily injective or surjective, we can interpret rank nullity as a statement about "information loss".

If an isomorphism, being invertible, perfectly preserves linear structure, a non-invertible map destroys some information about its pre-image: nontrivial linear combinations in the domain may become trivial in the image. In particular, you can imagine that the map crushes an entire subspace, the kernel, to a point in the image (since it maps everything in the kernel to $\mathbf{0}$).

But in a sense, this is the worst thing a linear map can do; the image is then a subspace with dimension equal to the rank, which is the difference between the domain's dimension and the kernel's dimension. The pre-image of a point will be an *affine subset* that looks like a translation of the kernel.

Examples

Example

Let A be a 10×7 matrix, and let B be a 10×12 matrix. What are the maximum possible ranks for A and B? What's the maximum possible rank of $\mathrm{A}^t\mathrm{B}$?

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Solution: The maximum rank of A is 7 and of B is 10. Since A^tB is a 7×12 matrix, its maximum possible rank is also 7.

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Example

Suppose a matrix A has 37 rows and 40 columns. What is the minimum dimension of $\operatorname{Nul}\mathrm{A}$?

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Example

Suppose a matrix A has 37 rows and 40 columns. What is the minimum dimension of $\mathrm{Nul}\,\mathrm{A}$?

Solution: We know that $\mathrm{rank}\,\mathrm{A} \leq 37$, and since $\mathrm{rank}\,\mathrm{A} + \mathrm{nullity}\,\mathrm{A} = 40$, we have that $\mathrm{nullity}\,\mathrm{A} \geq 40 - 37 = 3$. Thus the minimum $\dim \mathrm{Nul}\,\mathrm{A}$ is 3.

Examples

Example

If, for a 6×9 matrix A, every solution of the homogenous equation $\mathrm{A}\mathbf{x} = \mathbf{0}$ can be expressed as $r\mathbf{v}_1 + s\mathbf{v}_2 + t\mathbf{v}_3$ for linearly independent $\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3 \in \mathbb{R}^9$, then what can one say about existence of solutions to $\mathrm{A}\mathbf{x} = \mathbf{b}$ for any $\mathbf{b} \in \mathbb{R}^6$?

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Solution: We can conclude that $\mathrm{nullity}\,\mathrm{A} = 3$, and thus $\mathrm{rank}\,\mathrm{A} = 9 - 3 = 6$, so $\mathrm{Col}\,\mathrm{A} = \mathbb{R}^6$. Thus for every $\mathbf{b} \in \mathbb{R}^6$, there is an $\mathbf{x}$ such that $\mathrm{A}\mathbf{x} = \mathbf{b}$.

Examples

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Suppose a system of 3 equations in 4 unknowns has solutions for all possible choices of constants. Is it possible to find two solutions of the associated homogeneous system which are not multiples of each other?

Solution: All solutions of the homogeneous system will be multiples of each other: the system corresponds to a 3×4 matrix of rank 3, and consequently, the null space has dimension $4 - 3 = 1$, and is thus spanned by a single element.

Definition of Left Null Space

Definition

The *left null space* $\mathrm{LNul}\mathrm{A}$ of an $m \times n$ matrix A is the set of all $\mathbf{y} \in \mathbb{R}^m$ such that $\mathbf{y}^{\mathrm{t}}\mathrm{A} = \mathbf{0}$. Equivalently, it is the null space of A^{t}.

As with row space, one could define the left null space using row vectors (in this case, $1 \times m$ matrixes), in which case, as with row space, one would regard the left null space as distinct from $\mathrm{Nul}\,\mathrm{A}^{\mathrm{t}}$, but isomorphic to it.

The subspace $\mathrm{Nul}\,\mathrm{A}^{\mathrm{t}}$ of $\mathbb{R}^m$ is also sometimes called the *cokernel of the map* $\mathbf{x} \to \mathrm{A}\mathbf{x}$ (it is the kernel of the map $\mathbf{y} \to \mathrm{A}^{\mathrm{t}}\mathbf{y}$ from $\mathbb{R}^m$ to $\mathbb{R}^n$.)

Row Space and Left Null Space

Applying rank nullity to A^{t}, we see that

$$\dim \mathrm{Col}\,\mathrm{A}^{\mathrm{t}} + \dim \mathrm{Nul}\,\mathrm{A}^{t} = m\,,$$

whence, the dimension of the row space plus the dimension of the left null space sum to the number of rows of the matrix A:

$$\dim \mathrm{Row}\,\mathrm{A} + \dim \mathrm{LNul}\mathrm{A} = m.$$

Application: Graph Theory

The relationships between null space, column space, row space, and left null space appear in the subarea of combinatorics called *graph theory*.

By a graph $\mathcal{G}$, we mean a discrete (usually, but not always) finite set $\mathcal{V}$, called *vertices*, together with a collection $\mathcal{E}$ of unordered pairs of distinct vertices, called *edges*. One can visualize graphs as networks of nodes, possibly connected by line segments.

A *directed graph* has instead an edge set consisting of *ordered pairs* of distinct vertices. Their edges can be understood as having an assigned direction.

We can describe a finite directed graph $\mathcal{G}$ in two ways by a matrix: with an *adjacency matrix*, and with an *incidence matrix*. We'll discuss the latter briefly now, and the former later when we study eigenvalues and eigenvectors.

Application: Graph Theory

A directed edge $e = (v_j, v_k) \in \mathcal{E}$ is said to be *incident* to the vertices v_j and v_k and is said to "leave v_j" and to "enter v_k".

Definition

If for a finite graph $\mathcal{G} = (\mathcal{V}, \mathcal{E})$, $|\mathcal{V}| = m$ and $|\mathcal{E}| = n$, the *incidence matrix of* $\mathcal{G}$ is the $m \times n$ matrix $\mathrm{I}(\mathcal{G})$ whose (i, j)-th entry is equal to -1 if the j-th edge leaves the i-th vertex, $+1$ if the j-th edge enters the i-th vertex, and 0 if the j-th edge is not incident with the i-th vertex.

It is not uncommon for the incidence matrix to be defined in such a way as to be the transpose of our definition, in which case the roles of the null space and left null space discussed below are reversed.

Walks and Cycles

We define walks and cycles on graphs before we proceed to examine the row, column, null and left null spaces of an incidence matrix.

Definition

A walk on a graph is an alternating sequence of vertices and edges initiated and terminating in a vertex, with any consecutive vertex-edge or edge-vertex pair incident.

Thus, a walk can be specified by a sequence of *coincident* edges. A walk is called a *trail* if there are no repeated edges.

A *cycle* is a walk which returns to the initial vertex, and a *simple cycle* is a cycle which is also a trail.

Simple Cycles and Weights

If $\mathcal{G}$ is directed, then a simple cycle of $\mathcal{G}$ is associated to a vector $\mathbf{x} \in \mathbb{R}^n$, called a *weight vector*, with components $x_i \in \{+1, -1\}$, such that $\mathrm{I}(\mathcal{G})\mathbf{x} = \mathbf{0}$; an edge has positive weight if the cycle traverses it according to its orientation, and is negative if the edge is traversed against its orientation.

Simple cycles on a (not necessarily directed) graph are said to be independent if the corresponding weight vectors are linearly independent (for some orientations of edges).

Application: Graph Theory

Viewed as a matrix over $\mathbb{R}$, the subspaces associated to a graph's incidence matrix encode interesting *topological features*.

Claim

Let $\mathrm{I}(\mathcal{G})$ be the incidence matrix of a finite directed graph $\mathcal{G} = (\mathcal{V}, \mathcal{E})$ with m vertices and n edges.

1. The rank $\operatorname{rank} \mathrm{I}(\mathcal{G}) = m - k$ where k is the number of connected components of $\mathcal{G}$. In particular, for a connected graph, $k = 1$, and $\operatorname{rank} \mathrm{I}(\mathcal{G}) = m - 1$.
2. If $\mathcal{G}$ is connected, then its left null space is spanned by the vector $\mathbf{e}_1 + \mathbf{e}_2 + \ldots + \mathbf{e}_m$. In particular, the column space of $\mathrm{I}(\mathcal{G})$ is the set of all $\mathbf{x} \in \mathbb{R}^m$ perpendicular to $\mathbf{e}_1 + \mathbf{e}_2 + \ldots + \mathbf{e}_m$, and so the sum of components of such $\mathbf{x}$ must be zero.

Application: Graph Theory

Claim

3. The null space $\operatorname{Nul} \mathrm{I}(\mathcal{G})$ consists of vectors $\mathbf{x} \in \mathbb{R}^n$, whose components, thought of as currents on the corresponding edges, yield a solution to Kirchhoff's current law in the absence of a current source: in matrix form the law reads $A\mathbf{x} = \mathbf{0}$. In particular, the dimension of the null space is the maximum number of independent simple cycles in $\mathcal{G}$.

Challenge Problem: Prove the claim.

Application: Graph Theory

Let $\mathrm{I}(\mathcal{G})$ be the incidence matrix of a connected graph. Then applying the rank-nullity theorem, we obtain:
the maximum number of independent simple cycles $(= \mathrm{nullity}\, \mathrm{I}(\mathcal{G}))$ + the number of vertices -1 $(= \mathrm{rank}\, \mathrm{I}(\mathcal{G}))$ = the number of edges.

Rearranging:

max($\#$ of indep. simple cycles)$-$($\#$ of edges)$+$($\#$ of vertices) $= 1$

which is called *Euler's formula* for a connected graph.

Example/Exercises

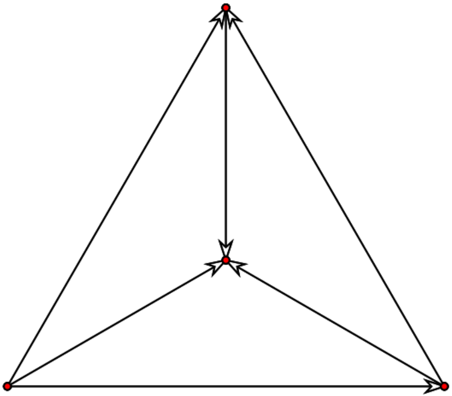

Figure: The directed graph $\mathcal{G} = \left(\{v_0, v_1, v_2, v_3\}, \{(v_0, v_1), (v_0, v_2), (v_0, v_3), (v_1, v_2), (v_1, v_3), (v_2, v_3)\}\right)$.

Exercises

Exercise

- For the figure above, write down an incidence matrix, and find bases for the column and null spaces.
- Verify by hand that there are at most 3 independent cycles by using the picture.
- What if the orientations are altered? Will there always be 3 independent cycles?
- What would an acyclic graph look like? Construct some graphs with 0, 1, 2, and 11 cycles, and check Euler's graph formula for each.

Definition of the Outer Product

Recall, the dot product of two vectors $\mathbf{u}, \mathbf{v} \in \mathbb{R}^n$ is given by $\mathbf{u} \cdot \mathbf{v} = \mathbf{u}^{\mathrm{t}}\mathbf{v}$. This is called an *inner product*.

The outer product of two n-vectors is a map from $\mathbb{R}^m \times \mathbb{R}^n \to \mathbb{R}^{m \times n}$:

Definition

Given two vectors $\mathbf{u} \in \mathbb{R}^m$ and $\mathbf{v} \in \mathbb{R}^n$, their outer product is the matrix $m \times n$

$$\mathbf{u} \otimes \mathbf{v} := \mathbf{u}\mathbf{v}^{\mathrm{t}} = \begin{bmatrix} u_1 v_1 & u_1 v_2 & \dots & u_1 v_n \\ u_2 v_1 & u_2 v_2 & \dots & u_2 v_n \\ \vdots & \vdots & \ddots & \vdots \\ u_m v_1 & u_m v_2 & \dots & u_m v_n \end{bmatrix}.$$

Rank One Matrices are Outer Products

Proposition

The rank of a matrix obtained via an outer product is one, and moreover any rank one real matrix can be realized as an outer product. That is, suppose $\mathrm{A} \in \mathbb{R}^{m\times n}$ *is a rank one matrix. Then there exist vectors* $\mathbf{r} \in \mathbb{R}^n$ *and* $\mathbf{c} \in \mathbb{R}^m$ *such that*

$$\mathbf{c} \otimes \mathbf{r} = \mathrm{A}\,.$$

Rank One Matrices are Outer Products

Proof.

We'll show that a rank one matrix is an outer product, and leave the fact that an outer product has rank one as an exercise.

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If $\mathrm{A} = \left[\begin{array}{ccc} \mathbf{a}_1 & \ldots & \mathbf{a}_n \end{array}\right]$ is a rank one matrix, then $\mathrm{Col}\,\mathrm{A}$ can be expressed as the span of a single of its nonzero column vectors, say $\mathbf{c} = \mathbf{a}_j \in \mathbb{R}^m$ for some j.

Rank One Matrices are Outer Products

Proof.

We'll show that a rank one matrix is an outer product, and leave the fact that an outer product has rank one as an exercise.

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The remaining columns, being also in the column space, are thus multiples of this column. Let r_k be a scalar such that $\mathbf{a}_k = r_k\mathbf{c}$. Then the vector $\mathbf{r} \in \mathbb{R}^n$ whose k-th component is r_k is a vector such that $\mathrm{A} = \mathbf{c} \otimes \mathbf{r}$. □

Example: Projection onto a Line

Example

Let $\mathbf{v} \in \mathbb{R}^n$. What is the geometric action of the map associated to the matrix $\frac{1}{\mathbf{v}\cdot\mathbf{v}}\mathbf{v} \otimes \mathbf{v}$?

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It is the matrix of orthogonal projection onto the line $\mathrm{Span}\left\{\mathbf{v}\right\}$.

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It is the matrix of orthogonal projection onto the line $\mathrm{Span}\,\{\mathbf{v}\}$.

In particular, if $\mathbf{u}$ is a unit vector spanning a line ℓ then $\mathrm{proj}_\ell(\mathbf{x}) = (\mathbf{u} \otimes \mathbf{u})\mathbf{x} = (\mathbf{u} \cdot \mathbf{x})\mathbf{u}$.

The Dual Space of a Vector Space

Definition

Let V be a vector space over a field $\mathbb{F}$. The *dual vector space* to V is the vector space V^* of *linear functionals* $f : V \to \mathbb{F}$.

In particular, note that any linear function on $\mathbb{R}^n$ has the form $f(x_1, \ldots, x_n) = a_1x_1 + \ldots + a_nx_n = \mathbf{a}^{\mathrm{t}}\mathbf{x}$. Thus, $(\mathbb{R}^n)^* = \mathbb{R}^{1\times n}$.

Thus, the row space of an $m \times n$ matrix A is best regarded as the subspace of $(\mathbb{R}^m)^* = \mathbb{R}^{1\times m}$.
Observe that the dual of a finite vector space is isomorphic to the original vector space (why?)

Homework

- Homework in MyMathLab for section 4.4 on coordinates is due Thursday, 4/3.
- Wednesday, 4/4, there will be a quiz on sections 4.3-4.6, due Monday, 4/9.
- Homework in MyMathLab for section 4.5 on dimension is due Tuesday, 4/5.
- **Exam 2 will be held Tuesday, April 4/10/18, 7:00PM-9:00PM, in Hasbrouck Lab Addition room 124.**

  The syllabus for the second midterm is the following sections of the textbook: 2.2, 2.3, 3.1, 3.2, 3.3, 4.1, 4.2, 4.3, 4.4, 4.5.