MATH 235-04, Spring 2018
Quiz 5 Solutions

1. Let $A = \begin{bmatrix} 7 & 2 & -3 & 0 & -21 \\ -2 & 4 & 10 & 1 & 10 \\ 3 & 0 & -3 & 0 & -9 \\ -1 & 3 & 7 & -5 & -17 \end{bmatrix}$ and let $S = \left\{ \begin{bmatrix} 1 \\ -2 \\ 1 \\ -1 \\ 0 \end{bmatrix}, \begin{bmatrix} 4 \\ 10 \\ -5 \\ -12 \\ 3 \end{bmatrix}, \begin{bmatrix} -5 \\ 4 \\ -2 \\ 4 \\ -1 \end{bmatrix} \right\}$.

(a) [8 pts] Which vectors in the set S are in the null space of A? What can you conclude about the minimum possible dimension of the null space?

(b) [5 pts] Based on the results of (a), can you conclude whether or not $A\mathbf{x} = \mathbf{b}$ has a solution for every $\mathbf{b} \in \mathbb{R}^4$? If you can draw a conclusion, do so with justification. Otherwise, state what additional information would be necessary to draw a conclusion.

(c) [12 pts] Find bases for Row A, Col A, and Nul A.

(a) Let $\mathbf{v}_1 = \begin{bmatrix} 1 \\ -2 \\ 1 \\ -1 \\ 0 \end{bmatrix}$, $\mathbf{v}_2 = \begin{bmatrix} 4 \\ 10 \\ -5 \\ -12 \\ 3 \end{bmatrix}$ and $\mathbf{v}_3 = \begin{bmatrix} -5 \\ 4 \\ -2 \\ 4 \\ -1 \end{bmatrix}$. Then after calculating matrix vector products one finds that $A\mathbf{v}_2 = \mathbf{0} = A\mathbf{v}_3$, while $A\mathbf{v}_1 = -\mathbf{a}_4$ where $A = \begin{bmatrix} \mathbf{a}_1 & \mathbf{a}_2 & \mathbf{a}_3 & \mathbf{a}_4 & \mathbf{a}_5 \end{bmatrix}$. In fact, $\mathbf{v}_1 = \mathbf{u}_1 - \mathbf{e}_4$ where $\mathbf{u}_1$ is the vector in part (c) below which is one of two vectors found that span the nullspace.

Thus, of the vectors in S, only $\mathbf{v}_2$ and $\mathbf{v}_3$ are in the null space of A. Note that $\mathbf{v}_1$ and $\mathbf{v}_2$ are not scalar multiples of each other, whence $\dim \operatorname{Nul} A \geq 2$.

(b) For $A\mathbf{x} = \mathbf{b}$ to have a solution for each $\mathbf{b} \in \mathbb{R}^4$, the rank must be 4. But since $\dim \operatorname{Nul} A \geq 2$, by rank-nullity, $\operatorname{rank} A \leq 5 - 2 = 3$. Hence, there exist vectors $\mathbf{b} \in \mathbb{R}^4$ for which no $\mathbf{x} \in \mathbb{R}^5$ satisfies $A\mathbf{x} = \mathbf{b}$.

(c) There are row equivalences

$$\begin{bmatrix} 7 & 2 & -3 & 0 & -21 \\ -2 & 4 & 10 & 1 & 10 \\ 3 & 0 & -3 & 0 & -9 \\ -1 & 3 & 7 & -5 & -17 \end{bmatrix} \sim \begin{bmatrix} 1 & -3 & -7 & 5 & 17 \\ 0 & -2 & -4 & 11 & 44 \\ 0 & 0 & 0 & 1 & 4 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & -1 & 0 & -3 \\ 0 & 1 & 2 & 0 & 0 \\ 0 & 0 & 0 & 1 & 4 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

So, e.g., one can take

$$\left\{ \begin{matrix} \begin{bmatrix} 1 & -3 & -7 & 5 & 17 \end{bmatrix} \\ \begin{bmatrix} 0 & -2 & -4 & 11 & 44 \end{bmatrix} \\ \begin{bmatrix} 0 & 0 & 0 & 1 & 4 \end{bmatrix} \end{matrix} \right\}$$

as a basis of Row A, and

$$\left\{ \begin{bmatrix} 7 \\ -2 \\ 3 \\ -1 \end{bmatrix}, \begin{bmatrix} 2 \\ 4 \\ 0 \\ 3 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \\ -5 \end{bmatrix} \right\}$$

as a basis of Col A. Finally, from the reduced row echelon form, we have that any $\mathbf{x}$ satisfying $A\mathbf{x} = \mathbf{0}$ can be written as

$$\mathbf{x} = \begin{bmatrix} s+3t \\ -2s \\ s \\ -4t \\ t \end{bmatrix} = s \underbrace{\begin{bmatrix} 1 \\ -2 \\ 1 \\ 0 \\ 0 \end{bmatrix}}_{\mathbf{u}_1} + t \underbrace{\begin{bmatrix} 3 \\ 0 \\ 0 \\ -4 \\ 1 \end{bmatrix}}_{\mathbf{u}_2},$$

whence a basis of Nul A is given by

$$\{\mathbf{u}_1, \mathbf{u}_2\} = \left\{ \begin{bmatrix} 1 \\ -2 \\ 1 \\ -1 \\ 0 \end{bmatrix}, \begin{bmatrix} 3 \\ 0 \\ 0 \\ -4 \\ 1 \end{bmatrix} \right\}.$$

Observe that the vectors $\mathbf{v}_2, \mathbf{v}_3 \in S$ given in part (a) satisfy $\mathbf{v}_2 = -5\mathbf{u}_1 + 3.\mathbf{u}_2$, $\mathbf{v}_3 = -2\mathbf{u}_1 - \mathbf{u}_2$. One could also, knowing that the rank is 3 (given there are 3 pivots in RREF(A)) and thus that null A $=$ dim Nul A $= 5 - 3 = 2$, take $\{\mathbf{v}_2, \mathbf{v}_3\}$ as a basis of Nul A, since it is a pair of linearly independent vectors in Nul A, and thus must span Nul A.

2. Let $\mathbb{P}_n := \{a_0 + a_1 t + \ldots + a_n t^n \mid a_0, \ldots, a_n \in \mathbb{R}\}$ be the space of real polynomials of degree at most n, where n is a nonnegative integer.

(a) [5 pts] Show that $\mathcal{B} = \{1 - t + t^2 - t^3, t - t^2 + t^3, t^2 - t^3, t^3\}$ is a basis of $\mathbb{P}_3$.

(b) [5pts] Let $\mathbf{p}(t) = 4t^3 + 3t^2 + 2t + 1$. Find the coordinate vector $[\mathbf{p}(t)]_{\mathcal{B}}$ of the polynomial $\mathbf{p}(t)$ in the basis $\mathcal{B}$.

(c) [15pts] **Challenge problem (for extra credit):**

Let $\mathcal{B}' = \{1 - t + t^2 - t^3 + t^4, t - t^2 + t^3 - t^4, t^2 - t^3 + t^4, t^3 - t^4, t^4\}$. You may assume this is a basis of $\mathbb{P}_4$. Let $\mathcal{I} : \mathbb{P}_3 \to \mathbb{P}_4$ be the map given by

$$\mathcal{I}[\mathbf{p}(t)] = \int_0^t \mathbf{p}(\tau)\, \mathrm{d}\tau\,.$$

Find a matrix $A_{\mathcal{B},\mathcal{B}'}$ representing the map $\mathcal{I}$ in the $\mathcal{B}$ and $\mathcal{B}'$ coordinates, i.e., find a matrix $A_{\mathcal{B},\mathcal{B}'}$ such that

$$[\mathcal{I}[\mathbf{p}(t)]]_{\mathcal{B}'} = A[\mathbf{p}(t)]_{\mathcal{B}}\,.$$

(a) Let $\mathbf{p}_1(t) = 1 - t + t^2 - t^3$, $\mathbf{p}_2(t) = t - t^2 + t^3$, $\mathbf{p}_3(t) = t^2 - t^3$, and $\mathbf{p}_4(t) = t^3$. Let $\mathcal{B}_S = (1, t, t^2, t^3)$ be the ordered standard monomial basis of $\mathbb{P}_3$. Then note that the matrix

$$\mathrm{B} = \left[\ \big[\mathbf{p}_1(t)\big]_{\mathcal{B}_S} \quad \big[\mathbf{p}_2(t)\big]_{\mathcal{B}_S} \quad \big[\mathbf{p}_3(t)\big]_{\mathcal{B}_S} \quad \big[\mathbf{p}_4(t)\big]_{\mathcal{B}_S}\ \right] = \begin{bmatrix} 1 & 0 & 0 & 0 \\ -1 & 1 & 0 & 0 \\ 1 & -1 & 1 & 0 \\ -1 & 1 & -1 & 1 \end{bmatrix}.$$

Since $\mathrm{RREF(B)} = \mathrm{I}_4$, the columns are linearly independent, whence the set $\mathcal{B}$ of polynomials is a linearly independent set. Since $\mathcal{B}$ is a set of four linearly independent polynomials in $\mathbb{P}_3$ and $\dim \mathbb{P}_3 = 4$, $\mathbb{P}_3 = \operatorname{span} \mathcal{B}$, so $\mathcal{B}$ forms a basis.

(b) To find $[\mathbf{p}(t)]_{\mathcal{B}}$ it is easiest to work instead with standard monomial coordinates. Note that for the matrix B in part (a) above, we have

$$\mathrm{B}\big[\mathbf{p}(t)\big]_{\mathcal{B}} = \big[\mathbf{p}(t)\big]_{\mathcal{B}_S} \implies \big[\mathbf{p}(t)\big]_{\mathcal{B}} = \mathrm{B}^{-1}\big[\mathbf{p}(t)\big]_{\mathcal{B}_S}.$$

One can equivalently solve the corresponding system which has augmented matrix

$$\left[\begin{array}{cccc|c} 1 & 0 & 0 & 0 & 1 \\ -1 & 1 & 0 & 0 & 2 \\ 1 & -1 & 1 & 0 & 3 \\ -1 & 1 & -1 & 1 & 4 \end{array}\right]$$

This yields a solution

$$[\mathbf{p}(t)]_{\mathcal{B}} = \begin{bmatrix} 1 \\ 3 \\ 5 \\ 7 \end{bmatrix},$$

which corresponds to the fact that $\mathbf{p}(t) = \mathbf{p}_1(t) + 3\mathbf{p}_2(t) + 5\mathbf{p}_3(t) + 7\mathbf{p}_4(t)$, as can be seen quickly upon observing that $1 = \mathbf{p}_1(t) + \mathbf{p}_2(t)$, $t = \mathbf{p}_2(t) + \mathbf{p}_3(t)$, $t^2 = \mathbf{p}_3(t) + \mathbf{p}_4(t)$ and $t^3 = \mathbf{p}_4(t)$. These (fairly obvious) relations correspond to the columns of the inverse matrix

$$\mathrm{B}^{-1} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 1 \end{bmatrix}.$$

(c) The challenge problem solutions are being withheld, so that students may continue to submit solutions. If you believe you have a solution, you may turn it in, or come present it to me in office hours, anytime before the last day of classes.