MATH 235-04, Spring 2018
Quiz 4 Solutions

1. Let $\mathcal{U}^{3\times 3} \subset \mathbb{R}^{3\times 3}$ denote the set of all upper triangular 3×3 matrices, which are 3×3 matrices $\mathrm{A} = (a_{ij})$ such that the entries a_{21}, a_{31} and a_{32} are all 0.

(a) [5 pts] Show that $\mathcal{U}^{3\times 3}$ is a vector subspace of the vector space $\mathbb{R}^{3\times 3}$ of 3×3 matrices.

(b) [5pts] Compute the dimension of $\mathcal{U}^{3\times 3}$ by exhibiting an explicit basis. For full credit you must argue that you your answer is a basis: show it spans $\mathcal{U}^{3\times 3}$ and is linearly independent.

(c) [5pts] Provide an explicit isomorphism from $\mathcal{U}^{3\times 3}$ to the subspace of lower triangular matrices $\mathcal{L}^{3\times 3}$. Note this is a subspace by arguments like those of part (a). For full credit you must argue that the map you provide is an isomorphism: show it is linear and invertible.

(a) Observe that $\mathcal{U}^{3\times 3} = \left\{ \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ 0 & a_{22} & a_{23} \\ 0 & 0 & a_{33} \end{bmatrix} \,\middle|\, a_{ij} \in \mathbb{R},\ i,j = 1,2,3 \right\}$.

Let $\mathrm{A} = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ 0 & a_{22} & a_{23} \\ 0 & 0 & a_{33} \end{bmatrix}$, $\mathrm{B} = \begin{bmatrix} b_{11} & b_{12} & b_{13} \\ 0 & b_{22} & b_{23} \\ 0 & 0 & b_{33} \end{bmatrix}$, and let $s \in \mathbb{R}$ be any scalar. Then note

$$s\mathrm{A} + \mathrm{B} = \begin{bmatrix} sa_{11} + b_{11} & sa_{12} + b_{12} & sa_{13} + b_{13} \\ 0 & sa_{22} + b_{22} & sa_{23} + b_{23} \\ 0 & 0 & sa_{33} + b_{33} \end{bmatrix} \in \mathcal{U}^{3\times 3}.$$

This shows that $\mathcal{U}^{3\times 3}$ is a subspace, by the subspace test. One could also directly verify that $\mathbf{0} \in \mathcal{U}^{3\times 3}$ and that $\mathcal{U}^{n\times n}$ is closed under addition and scaling, but our above test demonstrates these facts simultaneously.

(b) Let E_{ij} be the 3×3 matrix with all entries equal to zero, save the (i,j)-th entry, which is 1. Using matrix addition a generic matrix $\mathrm{A} \in \mathcal{U}^{3\times 3}$ can be expressed as

$$\mathrm{A} = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ 0 & a_{22} & a_{23} \\ 0 & 0 & a_{33} \end{bmatrix} = a_{11}\mathrm{E}_{11} + a_{12}\mathrm{E}_{12} + a_{13}\mathrm{E}_{13} + a_{22}\mathrm{E}_{22} + a_{23}\mathrm{E}_{23} + a_{33}\mathrm{E}_{33}\,.$$

This establishes that $\mathcal{U}^{3\times 3} = \mathrm{span}_{\mathbb{R}}\{\mathrm{E}_{ij} \,|\, i,j \in \{1,2,3\},\ i \leq j\}$.
The zero matrix $\mathbf{0}_{3\times 3}$ is uniquely expressed by setting all coefficients a_{ij}, $1 \leq i \leq j \leq 3$ equal to 0. Thus, the set $\mathcal{B}_{\mathcal{U}} := \{\mathrm{E}_{ij} \,|\, i,j \in \{1,2,3\},\ i \leq j\}$ is a basis of $\mathcal{U}^{3\times 3}$, and consequently we can conclude that $\dim \mathcal{U}^{3\times 3} = 6$.

Finally, I will remark that this idea easily generalizes to describe $\mathcal{U}^{n\times n}$ or $\mathcal{L}^{n\times n}$, the subspaces of upper, respectively, lower triangular $n \times n$ real matrices. E.g., for $\mathcal{U}^{n\times n}$ one obtains a basis

$$\mathcal{B}_{\mathcal{U}} = \{\mathrm{E}_{ij} \,|\, i,j \in \{1,\ldots,n\},\, i \leq j\}\ ,$$

from which we can conclude $\dim \mathcal{U}^{n\times n} = \frac{1}{2}n(n+1)$. Reversing the inequality on indices gives us a basis of the subspace of lower triangular matrices $\mathcal{L}^{n\times n}$.

(c) Observe that

$$\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ 0 & a_{22} & a_{23} \\ 0 & 0 & a_{33} \end{bmatrix}^{\mathrm{t}} = \begin{bmatrix} a_{11} & 0 & 0 \\ a_{12} & a_{22} & 0 \\ a_{13} & a_{23} & a_{33} \end{bmatrix} \in \mathcal{L}^{3\times 3}\,.$$

Since $(\mathrm{A}^{\mathrm{t}})^{\mathrm{t}} = \mathrm{A}$, this map is invertible. Thus, we only need to show that the transpose is linear to confirm that it provides an isomorphism $\mathcal{U}^{3\times 3} \cong \mathcal{L}^{3\times 3}$.
Generally writing $\mathrm{A} = (a_{ij})$, and $\mathrm{A}^{\mathrm{t}} = (a_{ji})$, observe that for any matrices A, B in $\mathbb{R}^{3\times 3}$ and any scalar $s \in \mathbb{R}$

$$(s\mathrm{A}+\mathrm{B})^{\mathrm{t}} = \big(s(a_{ij}) + b_{ij}\big) = \big((sa_{ij}) + (b_{ij})\big)^{\mathrm{t}}\,,$$

the (i,j)-th entry of which is the (j,i)-th entry of $\big((sa_{ij}) + (b_{ij})\big)$, namely, $\big((sa_{ji}) + (b_{ji})\big) = s\mathrm{A}^{\mathrm{t}} + \mathrm{B}^{\mathrm{t}}$, whence $(s\mathrm{A}+\mathrm{B})^{\mathrm{t}} = s\mathrm{A}^{\mathrm{t}} + \mathrm{B}^{\mathrm{t}}$. This establishes linearity of the transpose, and thus we have that $\bullet^{\mathrm{t}} : \mathcal{U}^{3\times 3} \to \mathcal{L}^{3\times 3}$ is a linear isomorphism with inverse the transpose map $\bullet^{\mathrm{t}} : \mathcal{L}^{3\times 3} \to \mathcal{U}^{3\times 3}$.
One could construct many other isomorphisms, e.g., by specifying any basis of $\mathcal{L}^{3\times 3}$ and then mapping the basis from part (b) onto the basis of $\mathcal{L}^{3\times 3}$, and extending by linearity. Observe that if one specifies a basis $\mathcal{B}_{\mathcal{L}}$ of $\mathcal{L}^{3\times 3}$ as a sub-basis of the standard basis $\mathcal{B}_S$ of $\mathbb{R}^{3\times 3}$, then by choosing the order appropriately in constructing the bijection $\mathcal{B}_{\mathcal{U}} \to \mathcal{B}_{\mathcal{L}}$, the associated isomorphism will just be the transpose map.