MATH 235-04, Spring 2018
Quiz 3 Solutions

1. Suppose $T : \mathbb{R}^3 \to \mathbb{R}^2$ is a linear map such that for some vectors $\mathbf{u}$ and $\mathbf{v}$

$$T(\mathbf{u}) = \begin{bmatrix} 1 \\ -1 \end{bmatrix}, \text{ and } T(2\mathbf{u} + 5\mathbf{v}) = \begin{bmatrix} 7 \\ -2 \end{bmatrix}.$$

(a) [5 pts] Find $T(\mathbf{v})$.

(b) [10pts] Given

$$\mathbf{u} = \begin{bmatrix} 2 \\ 1 \\ -2 \end{bmatrix} \text{ and } \mathbf{v} = \begin{bmatrix} -1 \\ 0 \\ 3 \end{bmatrix}$$

explicitly describe a map T meeting the above conditions by giving a matrix A with $T(\mathbf{x}) = \mathrm{A}\mathbf{x}$. Is your solution unique?

(a) Since T is linear, $T(2\mathbf{u} + 5\mathbf{v}) = 2T(\mathbf{u}) + 5T(\mathbf{v})$. Thus

$$\begin{aligned} T(\mathbf{v} &= \frac{1}{5}\left(\begin{bmatrix} 7 \\ -2 \end{bmatrix} - 2T(\mathbf{u})\right) = \frac{1}{5}\left(\begin{bmatrix} 7 \\ -2 \end{bmatrix} - \begin{bmatrix} 2 \\ -2 \end{bmatrix}\right) \\ &= \frac{1}{5}\begin{bmatrix} 7-2 \\ -2+2 \end{bmatrix} = \frac{1}{5}\begin{bmatrix} 5 \\ 0 \end{bmatrix} \\ &= \begin{bmatrix} 1 \\ 0 \end{bmatrix} = \mathbf{e}_1 . \end{aligned}$$

(b) Note that since T is a map from $\mathbb{R}^3$ to $\mathbb{R}^2$, a matrix A representing T must have dimensions 2×3. To find a matrix A with given values of $\mathbf{u}$ and $\mathbf{v}$ write

$$\mathrm{A} = \begin{bmatrix} a & b & c \\ d & e & f \end{bmatrix}.$$

Then

$$\mathrm{A}\mathbf{u} = \begin{bmatrix} a & b & c \\ d & e & f \end{bmatrix}\begin{bmatrix} 2 \\ 1 \\ -2 \end{bmatrix} = \begin{bmatrix} 1 \\ -1 \end{bmatrix},$$

$$\mathrm{A}\mathbf{v} = \begin{bmatrix} a & b & c \\ d & e & f \end{bmatrix}\begin{bmatrix} -1 \\ 0 \\ 3 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}.$$

This gives a system of 4 equations in 6 unknowns. Thus, there will not be a unique solution. The system is described by the augmented matrix

$$\left[\begin{array}{cccccc|c} 2 & 1 & -2 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 2 & 1 & -2 & -1 \\ -1 & 0 & 3 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & -1 & 0 & 3 & 0 \end{array}\right].$$

This reduces to the RREF matrix

$$\left[\begin{array}{cccccc|c} 1 & 0 & -3 & 0 & 0 & 0 & -1 \\ 0 & 1 & 4 & 0 & 0 & 0 & 3 \\ 0 & 0 & 0 & 1 & 0 & -3 & 0 \\ 0 & 0 & 0 & 0 & 1 & 4 & -1 \end{array}\right].$$

If $c = s$ and $f = t$ for some parameters $s, t \in \mathbb{R}$, then the matrix A must have the form

$$\mathrm{A}\mathbf{u} = \begin{bmatrix} 3s-1 & 3-4s & s \\ 3t & -1-4t & t \end{bmatrix}.$$

So, e.g., taking $s = 1$, $t = -1$, we get a linear transformation

$$T(\mathbf{x}) = \begin{bmatrix} 2 & -1 & 1 \\ -3 & 3 & -1 \end{bmatrix}\mathbf{x}.$$

There are of course infinitely many other possible transformations with the prescribed values for the given $\mathbf{u}$ and $\mathbf{v}$, but they all have the general form described above in terms of s and t.

2. A square $n \times n$ matrix which squares to itself is called an *idempotent*. An $n \times n$ matrix is a *nontrivial idempotent* if it is an idempotent which is neither the zero matrix nor the identity matrix. Let $\mathrm{A} \in \mathbb{R}^{n\times n}$ be a nontrivial idempotent matrix, so $\mathrm{A}^2 = \mathrm{A}$.

(a) Show that $\mathrm{I}_n - \mathrm{A}$ is also idempotent. Is $\mathrm{I}_n - \mathrm{A}$ invertible? Justify your answer carefully.

(b) Is the matrix $2\mathrm{A} - \mathrm{I}_n$ invertible? If so, find its inverse, expressed in terms of A.

(c) Find a 2×2 example of an idempotent matrix, none of the entries of which are zero.

(a) We check that $\mathrm{I}_n - \mathrm{A}$ squares to itself:

$$(\mathrm{I}_n - \mathrm{A})^2 = (\mathrm{I}_n - \mathrm{A})(\mathrm{I}_n - \mathrm{A}) = \mathrm{I}_n^2 - \mathrm{I}_n\mathrm{A} - \mathrm{A}\mathrm{I}_n + \mathrm{A}^2 = \mathrm{I}_n - 2\mathrm{A} + \mathrm{A} = \mathrm{I}_n - \mathrm{A}.$$

Now, notice that $(\mathrm{I}_n - \mathrm{A})\mathrm{A} = \mathrm{A} - \mathrm{A}^2 = \mathbf{0}_{n\times n}$, where $\mathbf{0}_{n\times n}$ is the $n\times n$ zero matrix. Therefore, since $\mathrm{A} \neq \mathbf{0}_{n\times n}$, it has at least one column which is a nontrivial solution of the equation $(\mathrm{I}_n - \mathrm{A})\mathbf{x} = \mathbf{0}$, whence $\mathrm{I}_n - \mathrm{A}$ is *not* invertible.
(b) The matrix $2\mathrm{A} - \mathrm{I}_n$ is invertible: it is its own inverse (such a matrix is said to be *involutive*). Indeed,

$$(2\mathrm{A} - \mathrm{I}_n)^2 = (2\mathrm{A} - \mathrm{I}_n)(2\mathrm{A} - \mathrm{I}_n) = 4\mathrm{A}^2 - 2\mathrm{A}\mathrm{I}_n - 2\mathrm{I}_n\mathrm{A} + \mathrm{I}_n = \mathrm{I}_n.$$

Thus $(2\mathrm{A} - \mathrm{I}_n)^{-1} = 2\mathrm{A} - \mathrm{I}_n$ by uniqueness of inverses.

(c) Any projection matrix for projection onto a line will be idempotent, since projecting onto a line preserves the line. Let θ be the angle made by a line ℓ with the x_1-axis. Recall, $\ell = \operatorname{span}\{\cos(\theta)\mathbf{e}_1 + \sin(\theta)\mathbf{e}_2\}$, and the matrix of the projection map is

$$\mathrm{P}_\ell = \begin{bmatrix} \cos^2(\theta) & \cos(\theta)\sin(\theta) \\ \cos(\theta)\sin(\theta) & \sin^2(\theta) \end{bmatrix}.$$

One can check that $\mathrm{P}_\ell^2 = \mathrm{P}_\ell$ using the Pythagorean identity $\cos^2(\theta) + \sin^2(\theta) = 1$. Thus, any 2×2 matrix of the form P_ℓ with θ not an integer multiple of $\pi/2$ will be an idempotent 2×2 matrix none of whose entries are 0. An example not of this form is

$$\begin{bmatrix} 2 & -1 \\ 2 & -1 \end{bmatrix}.$$

Geometrically, this still collapses $\mathbb{R}^2$ onto a line but it is not an orthogonal projection (it's a projection onto $\operatorname{span}\{\mathbf{e}_1 + \mathbf{e}_2\}$, along lines of slope 2).

Challenge problem: Describe matrix formulae for projection onto a line $\ell = \operatorname{span}\{\cos(\theta)\mathbf{e}_1 + \sin(\theta)\mathbf{e}_2\}$ along lines of slope m for any value of $m \neq \tan(\theta)$ (including undefined m/projection along vertical lines when applicable). Do the resulting matrices account for all possible 2×2 nontrivial idempotents? Justify your claim.

3. The matrix

$$\mathrm{M} = \begin{bmatrix} 1 & a & a^2 & a^3 \\ 1 & b & b^2 & b^3 \\ 1 & c & c^2 & c^3 \\ 1 & d & d^2 & d^3 \end{bmatrix}$$

is an example of a *Vandermonde matrix.*

(a) [10 pts] Find det(M) by row reduction. What conditions on the quantities a, b, c and d ensure that M is invertible?

(b) [5 pts] Describe a linear system, the solution of which gives the coefficients of a cubic polynomial whose graph passes through the points $(1, -2)$, $(2, 5)$, $(3, 20)$, and $(4, 49)$. Solve the system.

(c) [5 pts] Use the solution of part (a) of this question to compute the determinant of the coefficient matrix from part (b).

(a) The calculation proceeds via the following three pivoting sequences of row operations:

$$(i)\ \begin{cases} R_2 - R_1 \mapsto R_2, \\ R_3 - R_1 \mapsto R_3, \\ R_4 - R_1 \mapsto R_4, \end{cases} \qquad (ii)\ \begin{cases} R_3 - \frac{c-a}{b-a} R_2 \mapsto R_3, \\ R_4 - \frac{d-a}{b-a} R_2 \mapsto R_4, \end{cases}$$

$$(iii)\ \left\{R_4 - \tfrac{(d-a)(d-b)}{(c-a)(c-b)} R_3 \mapsto R_4\,.\right.$$

The pivotal moments (pun intended) in the sequence of resulting row equivalent matrices are shown below.

$$\begin{bmatrix} 1 & a & a^2 & a^3 \\ 1 & b & b^2 & b^3 \\ 1 & c & c^2 & c^3 \\ 1 & d & d^2 & d^3 \end{bmatrix} \overset{(i)}{\longmapsto} \begin{bmatrix} 1 & a & a^2 & a^3 \\ 0 & b-a & b^2-a^2 & b^3-a^3 \\ 0 & c-a & c^2-a^2 & c^3-a^3 \\ 0 & d-a & d^2-a^2 & d^3-a^3 \end{bmatrix}$$

$$\overset{(ii)}{\longmapsto} \begin{bmatrix} 1 & a & a^2 & a^3 \\ 0 & b-a & b^2-a^2 & b^3-a^3 \\ 0 & 0 & (c-a)(c-b) & (c-a)(c-b)(a+b+c) \\ 0 & 0 & (d-a)(d-b) & (d-a)(d-b)(a+b+d) \end{bmatrix}$$

$$\overset{(iii)}{\longmapsto} \begin{bmatrix} 1 & a & a^2 & a^3 \\ 0 & b-a & b^2-a^2 & b^3-a^3 \\ 0 & 0 & (c-a)(c-b) & (c-a)(c-b)(a+b+c) \\ 0 & 0 & 0 & (d-a)(d-b)(d-c) \end{bmatrix}.$$

Since the elementary row operations employed did not involve row swaps or rescaling, the determinant of the final matrix equals the determinant of the original 4×4 Vandermonde matrix. The final matrix being upper triangular, we deduce that the determinant is the product of the diagonal entries, so

$$\det(M) = (b-a)(c-a)(c-b)(d-a)(d-b)(d-c)\,.$$

It is clear that the determinant is nonzero if and only if a, b, c and d are all distinct, and so M is invertible if and only if a, b, c, and d are all distinct.

(b) Write the cubic polynomial as $p(x) = a_0 + a_1 x + a_2 x^2 + a_3 x^3$. Then, from the given points, we obtain four linear equations:

$$\begin{cases} a_0 + \ a_1 + \ a_2 + \ a_3 & = -2 \\ a_0 + 2a_1 + 4a_2 + 8a_3 & = 5 \\ a_0 + 3a_1 + 9a_2 + 27a_3 & = 20 \\ a_0 + 4a_1 + 16a_2 + 64a_3 & = 49 \end{cases}$$

This has the augmented matrix

$$\left[\begin{array}{cccc|c} 1 & 1 & 1 & 1 & -2 \\ 1 & 2 & 4 & 8 & 5 \\ 1 & 3 & 9 & 27 & 20 \\ 1 & 4 & 16 & 64 & 49 \end{array}\right].$$

The coefficient matrix is of course a 4×4 Vandermonde matrix, as encountered in part (a). We can use the row operations described in the solution to (a) to reduce to an upper triangular matrix, and then pivot up/back-substitute to arrive at the reduced row echelon form. Doing this, we find that the polynomial is

$$p(x) = -7 + 6x - 2x^2 + x^3\,.$$

(c) The determinant of the coefficient matrix is just the product of differences of the x-coordinates of the given points ordered appropriately:

$$(b-a)(c-a)(c-b)(d-a)(d-b)(d-c)$$

with $a = 1$, $b = 2$, $c = 3$ and $d = 4$. Thus

$$\begin{vmatrix} 1 & 1 & 1 & 1 \\ 1 & 2 & 4 & 8 \\ 1 & 3 & 9 & 27 \\ 1 & 4 & 16 & 64 \end{vmatrix} = (2-1)(3-1)(3-2)(4-1)(4-2)(4-3) = 12\,.$$