MATH 235-04, Spring 2018
Quiz 2 Solutions

1. Let A be the matrix of the reflection of $\mathbb{R}^2$ through the line $\ell = \operatorname{span}\left\{\begin{bmatrix} 2 \\ 1 \end{bmatrix}\right\}$, and let B be the matrix of reflection through the x_1-axis. Find the matrix of the linear map which takes any vector $\mathbf{x} \in \mathbb{R}^2$ to the vector obtained by first reflecting $\mathbf{x}$ through the x_1-axis, and then reflecting the result through ℓ.
What is the geometric interpretation of this map?

The matrix A of the map $\mathrm{Ref}_\ell : \mathbb{R}^2 \to \mathbb{R}^2$ was found in class to be

$$\mathrm{A} = \begin{bmatrix} 3/5 & 4/5 \\ 4/5 & -3/5 \end{bmatrix}.$$

The matrix B of the map $\mathrm{Ref}_{\operatorname{span}\mathbf{e}_1}$ is

$$\mathrm{B} = \begin{bmatrix} \mathrm{Ref}_{\operatorname{span}\mathbf{e}_1}(\mathbf{e}_1) & \mathrm{Ref}_{\operatorname{span}\mathbf{e}_1}(\mathbf{e}_2) \end{bmatrix} = \begin{bmatrix} \mathbf{e}_1 & -\mathbf{e}_2 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}.$$

The map whose matrix we are to compute is the composition $\mathrm{Ref}_\ell \circ \mathrm{Ref}_{\operatorname{span}\mathbf{e}_1} : \mathbb{R}^2 \to \mathbb{R}^2$, whence the matrix is given by

$$\mathrm{AB} = \begin{bmatrix} 3/5 & 4/5 \\ 4/5 & -3/5 \end{bmatrix}\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} = \begin{bmatrix} 3/5 & -4/5 \\ 4/5 & 3/5 \end{bmatrix}.$$

This matrix is a rotation matrix, and geometrically, the map $\mathrm{Ref}_\ell \circ \mathrm{Ref}_{\operatorname{span}\mathbf{e}_1} : \mathbb{R}^2 \to \mathbb{R}^2$ is a counterclockwise rotation of $\mathbb{R}^2$ by twice the angle $\varphi = \arccos(2/\sqrt{5})$ between the x_1 axis and ℓ. Thus, it is rotation by the angle $\theta = 2\varphi = \arccos(3/5)$. To see this, note that the columns of M are each unit vectors whose components satisfy

$$\begin{aligned} \frac{3}{5} &= \left(\frac{2}{\sqrt{5}}\right)^2 - \left(\frac{1}{\sqrt{5}}\right)^2 = \cos^2\varphi - \sin^2\varphi = \cos(2\varphi) \\ \pm\frac{4}{5} &= \pm 2\left(\frac{2}{\sqrt{5}}\right)\left(\frac{1}{\sqrt{5}}\right) = \pm 2\sin\varphi\cos\varphi = \pm\sin(2\varphi). \end{aligned}$$

Compare this to the general form of a counterclockwise rotation matrix for an angle $\theta > 0$:

$$\mathrm{R}_\theta = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}.$$

Setting $\theta = 2\varphi$ gives M for φ as described.
Generally, a composition of two reflections about a pair of lines meeting at $\mathbf{0}$ with angle of separation φ will produce either a clockwise or counterclockwise rotation by the angle 2θ. Whether it is clockwise or counterclockwise depends on which line is the first to be reflected over, and which of the vertical angle pairs is chosen to correspond to θ; one can always realize a counterclockwise rotation by $\theta = 2\varphi$ as a clockwise rotation by $2\pi - \theta = 2(\pi - \varphi)$.

2. For the matrix obtained in the preceding question representing the composition of the two reflections, find the inverse matrix. Call it M, and compute

$$\mathrm{M}\begin{bmatrix} 1 \\ 2 \end{bmatrix}.$$

The inverse can be computed by the formula

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix}^{-1} = \frac{1}{ad-bc}\begin{bmatrix} d & -b \\ -c & a \end{bmatrix},$$

if it exists. Thus,

$$\mathrm{M} = (\mathrm{AB})^{-1} = \frac{1}{(3/5)(3/5) - (4/5)(-4/5)}\begin{bmatrix} 3/5 & 4/5 \\ -4/5 & 3/5 \end{bmatrix} = \begin{bmatrix} 3/5 & -4/5 \\ 4/5 & 3/5 \end{bmatrix}.$$

Knowing what we now know, we also can compute M knowing that both A and B are reflection matrices and thus each is self-inverse, together with the property that $(\mathrm{AB})^{-1} = \mathrm{B}^{-1}\mathrm{A}^{-1}$. Thus, $\mathrm{M} = \mathrm{B}^{-1}\mathrm{A}^{-1} = \mathrm{BA}$.

M is also a rotation matrix, but it is a *clockwise rotation* by twice the angle of separation φ between the x_1-axis and ℓ.

Finally $\mathrm{M}\begin{bmatrix} 1 \\ 2 \end{bmatrix} = \begin{bmatrix} 3/5 + 8/5 \\ -4/5 + 6/5 \end{bmatrix} = \begin{bmatrix} 11/5 \\ 2/5 \end{bmatrix}.$