MATH 235-04, Spring 2018
Quiz 1 Solutions

1. For the vectors below, construct an equation of the form $A\mathbf{x} = \mathbf{b}$. Then solve for $\mathbf{x}$ and exhibit $\mathbf{b}$ as a linear combination of the vectors $\mathbf{a}_1$, $\mathbf{a}_2$, and $\mathbf{a}_3$ (which may be regarded as the columns of the matrix A).

$$\mathbf{a}_1 = \begin{bmatrix} 0 \\ -1 \\ 1 \end{bmatrix} \qquad \mathbf{a}_2 = \begin{bmatrix} -1 \\ 1 \\ -1 \end{bmatrix} \qquad \mathbf{a}_2 = \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix} \qquad \mathbf{b} = \begin{bmatrix} 2 \\ -3 \\ -2 \end{bmatrix}$$

You must list or show any row operations performed for full credit, and clearly mark your final answer giving $\mathbf{b}$ in terms of the vectors $\mathbf{a}_1$, $\mathbf{a}_2$, and $\mathbf{a}_3$.

The matrix A will have $\mathbf{a}_1$, $\mathbf{a}_2$, and $\mathbf{a}_3$ as its columns, hence we can write a matrix vector equation of the form

$$A\mathbf{x} = \begin{bmatrix} 0 & -1 & 1 \\ -1 & 1 & -1 \\ 1 & -1 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 2 \\ -3 \\ -2 \end{bmatrix} = \mathbf{b}\,.$$

We can convert this into a system:

$$\begin{bmatrix} 0 & -1 & 1 \\ -1 & 1 & -1 \\ 1 & -1 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} -x_2 + x_3 \\ -x_1 + x_2 - x_3 \\ x_1 - x_2 \end{bmatrix} = \begin{bmatrix} 2 \\ -3 \\ -2 \end{bmatrix}$$

$$\iff \begin{cases} 0x_1 - x_2 + x_3 &= 2 \\ -x_1 + x_2 - x_3 &= -3 \\ x_1 - x_2 + 0x_3 &= -2 \end{cases}.$$

This corresponds to an augmented matrix

$$\left[\mathrm{A} \mid \mathbf{b}\right] = \left[\begin{array}{ccc|c} 0 & -1 & 1 & 2 \\ -1 & 1 & -1 & -3 \\ 1 & -1 & 0 & -2 \end{array}\right].$$

We can row reduce this by the following series of row operations:

1. $R_1 \leftrightarrow R_3$
2. $R_2 + R_1 \mapsto R_2$
3. $R_2 \leftrightarrow R_3$
4. $-R_2 \mapsto R_2$
5. $-R_3 \mapsto R_3$
6. $R_2 + R_3 \mapsto R_2$
7. $R_1 + R_2 \mapsto R_1$

This yields

$$\mathrm{RREF}\left(\left[\mathrm{A} \mid \mathbf{b}\right]\right) = \left[\begin{array}{ccc|c} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 3 \\ 0 & 0 & 1 & 5 \end{array}\right],$$

which implies that

$$\mathbf{x} = \begin{bmatrix} 1 \\ 3 \\ 5 \end{bmatrix},$$

and

$$\mathbf{b} = \mathrm{A}\mathbf{x} = \mathbf{a}_1 + 3\mathbf{a}_2 + 5\mathbf{a}_3\,,$$

which is easily checked using the rules for vector scaling and vector addition.

Observe that one could reorder the columns of A in order to avoid some of the row swaps (since the order in which vectors appear in the linear combination constructing **b** doesn't matter, since vector addition is commutative), but then one has to be careful to assign the correct scalar weights to each of $\mathbf{a}_1$, $\mathbf{a}_2$, and $\mathbf{a}_3$ when writing **b** as a linear combination of $\mathbf{a}_1$, $\mathbf{a}_2$, and $\mathbf{a}_3$ from the solution (x_1, x_2, x_3) to the system with the permuted columns.