Linear Dependence and Independence

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Outline

Nontrivial Linear Combinations Equal to $\mathbf{0}$

Reconsider the meaning of a nontrivial solution $\mathbf{x} \neq \mathbf{0} \in \mathbb{R}^n$ to a homogeneous system

$$Ax = \mathbf{0} \in \mathbb{R}^m .$$

If the columns of A are $\mathbf{a}_1, \ldots, \mathbf{a}_n$, then this means there is a linear combination

$$x_1\mathbf{a}_1 + \ldots + x_n\mathbf{a}_n = \mathbf{0} ,$$

where at least one of the coefficients x_i is nonzero.

Now, this could be uninteresting (e.g. if the matrix A is full of only zeroes), but we know of examples of nontrivial solutions to nontrivial systems.

An Example with Nontrivial Solutions

For example, the system

$$\underbrace{\begin{bmatrix} 2 & 3 & -1 \\ 1 & 2 & 1 \\ 4 & 5 & -5 \end{bmatrix}}_{A} \underbrace{\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}}_{\mathbf{x}} = \mathbf{0}$$

has infinitely many solutions, corresponding to the line of intersection of the three planes with equations $2x_1 + 3x_2 - x_3 = 0$, $x_1 + 2x_2 + x_3 = 0$, and $4x_1 + 5x_2 - 5x_3 = 0$.

An Example with Nontrivial Solutions

Indeed, after performing Gauss-Jordan, one obtains that

$$\left[\begin{array}{ccc|c} 2 & 3 & -1 & 0 \\ 1 & 2 & 1 & 0 \\ 4 & 5 & -5 & 0 \end{array}\right] \sim \left[\begin{array}{ccc|c} 1 & 0 & -5 & 0 \\ 0 & 1 & 3 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right],$$

which leaves the variable z free, and gives equations $x - 5z = 0$ and $y + 3z = 0$.
Thus,

$$\mathbf{x} = \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 5z \\ -3z \\ z \end{bmatrix} = z \begin{bmatrix} 5 \\ -3 \\ 1 \end{bmatrix}.$$

An Example with Nontrivial Solutions

Write $z = t$ and let

$$\mathbf{v} = \begin{bmatrix} 5 \\ -3 \\ 1 \end{bmatrix}.$$

We can thus describe the solution to this system as the set

$$\{t\mathbf{v} \mid t \in \mathbb{R}\} = \operatorname{Span}\{\mathbf{v}\}.$$

Observe that for any $t \in \mathbb{R}$ the columns of the original matrix A thus satisfy

$$5t\begin{bmatrix} 2 \\ 1 \\ 4 \end{bmatrix} - 3t\begin{bmatrix} 3 \\ 2 \\ 5 \end{bmatrix} + t\begin{bmatrix} -1 \\ 1 \\ -5 \end{bmatrix} = \mathbf{0}.$$

A Linear Dependency Relation

So long as $t \neq 0$, we get a relation between these columns: we can in particular write the last one as a linear combination of the first two:

$$\begin{bmatrix} -1 \\ 1 \\ -5 \end{bmatrix} = 3\begin{bmatrix} 3 \\ 2 \\ 5 \end{bmatrix} - 5\begin{bmatrix} 2 \\ 1 \\ 4 \end{bmatrix}.$$

Our example demonstrates that a nontrivial solution to the homogeneous equation $\mathrm{A}\mathbf{x} = \mathbf{0}$ yields a linear combination of the columns of A that gives the zero vector, from which we deduce that the columns exhibit a *linear dependency relation*.

This motivates us to introduce concepts of linear dependence and independence between vectors.

The Trivial Way to $\mathbf{0}$

Let $\mathbf{v}_1, \ldots, \mathbf{v}_p$ be a collection of vectors in $\mathbb{R}^n$. One can obtain the zero vector $\mathbf{0} \in \mathbb{R}^n$ as a linear combination of $\mathbf{v}_1, \ldots, \mathbf{v}_p$ in the obvious way:

$$0\mathbf{v}_1 + 0\mathbf{v}_2 + \ldots + 0\mathbf{v}_p = \mathbf{0}\,.$$

This is called the *trivial combination* of $\mathbf{v}_1, \ldots, \mathbf{v}_p$.

We are of course interested in *nontrivial combinations* of $\mathbf{v}_1, \ldots, \mathbf{v}_p$ giving $\mathbf{0}$, which are combinations

$$\sum_{i=1}^{p} x_i \mathbf{v}_i = \mathbf{0}$$

where are least one $x_i \neq 0$.

This is just a reframing of the question "does a nontrivial solution to $[\mathbf{v}_1 \ \ldots \mathbf{v}_p]\mathbf{x} = \mathbf{0}$ exist?"

Defining Linear Dependence

Definition

A collection of vectors $\mathbf{v}_1, \ldots, \mathbf{v}_p$ in $\mathbb{R}^n$ is called *linearly dependent* if there exists a nontrivial linear combination

$$\sum_{i=1}^{p} x_i \mathbf{v}_i = x_1\mathbf{v}_1 + \ldots + x_p\mathbf{v}_p = \mathbf{0}$$

with at least one coefficient x_i nonzero.

Nontrivial Solutions $\iff$ Linear Dependence

Observation

A set $\mathbf{v}_1, \ldots, \mathbf{v}_p$ of vectors in $\mathbb{R}^n$ is linearly dependent if and only if the matrix $[\mathbf{v}_1 \ \ldots \ \mathbf{v}_p]$ has fewer than p pivot positions, since the homogeneous equation $[\mathbf{v}_1 \ \ldots \ \mathbf{v}_p]\mathbf{x} = \mathbf{0}$ must have a nontrivial solution (and therefore, there is at least one free variable and infinitely many solutions).

Dependence Relations

Definition

A nontrivial linear relation $x_1\mathbf{v}_1 + \ldots + x_p\mathbf{v}_p = \mathbf{0}$ with at least one coefficient x_i nonzero is called a *linear dependency relation for the set of vectors* $\mathbf{v}_1, \ldots, \mathbf{v}_p$.

Observation

Suppose a linear dependency relation $x_1\mathbf{v}_1 + \ldots + x_p\mathbf{v}_p = \mathbf{0}$, and without loss of generality (by reordering/relabeling), assume $x_p \neq 0$. Then

$$\mathbf{v}_p = -\frac{x_1}{x_p}\mathbf{v}_1 - \ldots - \frac{x_{p-1}}{x_p}\mathbf{v}_{p-1} = -\sum_{i=1}^{p-1} \frac{x_i}{x_p}\mathbf{v}_i\,.$$

Thus a linear dependency relation implies one of the vectors is a linear combination of the others.

Dependence and Solutions

A relation expressing a vector $\mathbf{b}$ as a linear combination of $\mathbf{v}_1, \ldots, \mathbf{v}_p$ gives rise to a linear dependency relation

$$\mathbf{b} - \sum_{i=1}^{p} x_i \mathbf{v}_i = \mathbf{0}$$

on the collection $\mathbf{b}, \mathbf{v}_1, \ldots, \mathbf{v}_p$.

Correspondingly the system $[\mathbf{v}_1 \ \ldots \ \mathbf{v}_p]\mathbf{x} = \mathbf{b}$ is solvable precisely if there is a nontrivial solution $\mathbf{y} \in \mathbb{R}^{p+1}$ of the homogeneous system $[\mathbf{v}_1 \ \ldots \mathbf{v}_p \ \mathbf{b}]\mathbf{y} = \mathbf{0}$ with $y_{p+1} \neq 0$.

Independent Vectors

The complimentary definition captures when a set of vectors have no nontrivial linear relations to each other.

Definition

A collection of vectors $\mathbf{v}_1, \ldots, \mathbf{v}_p$ in $\mathbb{R}^n$ is said to be linearly independent if the only linear combination

$$x_1\mathbf{v}_1 + \ldots + x_p\mathbf{v}_p$$

equal to the zero vector is the trivial one.
That is, $\{\mathbf{v}_1, \ldots, \mathbf{v}_p\} \subset \mathbb{R}^n$ is a linearly independent set if and only if the following implication holds:

$$x_1\mathbf{v}_1 + \ldots + x_p\mathbf{v}_p = \mathbf{0} \implies x_i = 0 \text{ for all } i = 1, \ldots, p\,.$$

Criteria for Independence

Observation

A set of vectors $\mathbf{v}_1, \ldots, \mathbf{v}_p$ in $\mathbb{R}^n$ is linearly independent if and only if the equation

$$[\mathbf{v}_1 \ \ldots \ \mathbf{v}_p]\mathbf{x} = \mathbf{0}$$

is uniquely solved by the zero vector $\mathbf{x} = \mathbf{0} \in \mathbb{R}^p$. Thus the matrix must have precisely $p \leq n$ pivot positions, since there can be no free variables.

Remark

These conditions guarantee that no vector $\mathbf{v}_i$ in a linearly independent set can be written as a linear combination of the other vectors in the set.

Independence and Spans

An idea we will come back to is that linearly independent sets are *minimal generating sets* for their spans.

What is meant by this is the following. Consider a finite set S of vectors in $\mathbb{R}^n$. Suppose $\mathbf{v} \in S$ is a vector such that

$$\mathrm{Span}\, S = \mathrm{Span}\,(S - \{\mathbf{v}\})\,.$$

Then S is a linearly *dependent* set. On the other hand, if a finite set S' also containing $\mathbf{v}$ is linearly *independent*, then

$$\mathrm{Span}\,(S' - \{\mathbf{v}\}) \subsetneq \mathrm{Span}\, S'\,,$$

that is, removing $\mathbf{v}$ reduces the span to a "smaller" set (in this case, a set of smaller "dimension", which is a notion we will define later).

A challenge problem: Justify (i.e. prove) the statements of the preceding slide.

Moreover, describe an algorithm to reduce a linearly dependent set $S \subset \mathbb{R}^n$ of finitely many vectors to a linearly independent set $S' \subset \mathbb{R}^n$ such that $\mathrm{Span}\, S = \mathrm{Span}\, S'$.

Then show that regardless of any choices made in the algorithm, the final number of vectors in S' will be the same, and depends only on $\mathrm{Span}\, S$ itself (and not on S or choices you made).

Finally, explain why the number $|S'|$ of vectors in the linearly independent set S' must be less than or equal to n.

Dependence and Independence for a Pair of Vectors

If a pair $\mathbf{u}, \mathbf{v} \in \mathbb{R}^n$ of vectors is linearly dependent, then there are numbers a, b at least one of which is nonzero, such that $a\mathbf{u} + b\mathbf{v} = \mathbf{0}$.

Suppose $a \neq 0$. Then $\mathbf{u} = \frac{b}{a}\mathbf{v}$. If $a = 0$ then $b \neq 0$, so $\mathbf{v} = (0/b)\mathbf{u} = \mathbf{0}$. Similarly, one can show that either $\mathbf{v} = \frac{a}{b}\mathbf{u}$ or $\mathbf{v} = \mathbf{0}$.

Thus *a pair of vectors is linearly dependent if and only if one of them is a multiple of the other*.

We conclude that if neither $\mathbf{u}$ nor $\mathbf{v}$ is the zero vector, $\mathrm{Span}\,\{\mathbf{u}\} = \mathrm{Span}\,\{\mathbf{v}\}$, so geometrically these vectors are *collinear*, as they live on the same line in $\mathbb{R}^n$ through the origin. If one of them is the zero vector, they are still collinear, but their spans are no longer the same.

Dependence/Independence for a Vector Triple

A pair of vectors which are *independent* cannot be collinear, and will span a plane.

A vector triple is independent if the three vectors do not all lie in a common plane through the origin. A triple of dependent vectors will be *coplanar* (possibly collinear).

Example

Are the vectors

$$\mathbf{v}_1 = \begin{bmatrix} 0 \\ \sqrt{2} \\ 0 \end{bmatrix}, \quad \mathbf{v}_2 = \begin{bmatrix} e \\ 0 \\ -e \end{bmatrix} \quad \mathbf{v}_3 = \begin{bmatrix} -\pi \\ 1 \\ \pi \end{bmatrix}$$

linearly independent?

An Example: Coplanar?

Example

They are not linearly independent. In this case you might even guess a linear dependency relation, such as:

$$\begin{bmatrix} -\pi \\ 1 \\ \pi \end{bmatrix} = \frac{1}{\sqrt{2}} \begin{bmatrix} 0 \\ \sqrt{2} \\ 0 \end{bmatrix} - \frac{\pi}{e} \begin{bmatrix} e \\ 0 \\ -e \end{bmatrix} .$$

Another way to check is to row reduce the matrix

$$\begin{bmatrix} 0 & e & -\pi \\ \sqrt{2} & 0 & 1 \\ 0 & -e & \pi \end{bmatrix} .$$

You don't need RREF; row echelon form will do.

2 Pivot Positions for a $3 \times 3 \implies$ a Coplanar Triple

Example

Swapping the first and second rows gives

$$\begin{bmatrix} \sqrt{2} & 0 & 1 \\ 0 & e & -\pi \\ 0 & -e & \pi \end{bmatrix},$$

and then replacing the third row by its sum with the second gives a row of zeros.

We see that there are pivots in the $(1,1)$ and $(2,2)$ positions. The homogeneous system would have a free variable. If we instead think of this as an augmented matrix and continue to RREF, then we exhibit the last vector as a linear combination of the first two.

Thus, we've shown that vectors $\mathbf{v}_1$, $\mathbf{v}_2$, and $\mathbf{v}_3$ are coplanar.

Independence and Pivot Positions: an example

Example

On the other hand if we replace $\mathbf{v}_3$ by $\mathbf{v}_3' = \begin{bmatrix} \pi \\ 1 \\ \pi \end{bmatrix}$, the triple $\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3'$ *is linearly independent*. You can check that

$$\mathrm{RREF}\begin{bmatrix} 0 & e & \pi \\ \sqrt{2} & 0 & 1 \\ 0 & -e & \pi \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

Since the matrix of these three vectors has three pivot positions, they are linearly independent, and span $\mathbb{R}^3$.

Dependence for Sets of ≥ 3 Vectors

Theorem

An ordered collection $(\mathbf{v}_1, \ldots, \mathbf{v}_p)$ of two or more vectors is linearly dependent if and only if at least one of the vectors is a linear combination of the others. Moreover, if $\mathbf{v}_1 \neq \mathbf{0}$, then there is some $i > 1$ such that $\mathbf{v}_i$ is a linear combination of $\mathbf{v}_1, \ldots, \mathbf{v}_{i-1}$.

Remark

It is not necessarily true that each $\mathbf{v}_j$ is a linear combination of the others; just that there is *some* vector which is a linear combination of its predecessors (provided one of the predecessors isn't the zero vector).

Sets Containing $\mathbf{0}$

Remark

We need the caveats about zero vectors because otherwise we can find contradictions.

For example, consider an ordered collection consisting of two copies of the zero vector followed by some nonzero vector.

This set is linearly dependent (why?), but the last vector is *not* a linear combination of the other two! You can't get a nonzero vector by combining the zero vector with itself.

Sets Containing $\mathbf{0}$

More generally, we have:

Proposition

If a set of vectors contains the zero vector, then it is linearly dependent.

Proof.

One times the zero vector summed with zeroes times the other vectors in the set gives a linear combination equal to the zero vector, with a nonzero coefficient. □

- A set is $\mathbf{v}_1, \ldots, \mathbf{v}_p$ linearly *independent* $\iff$ whenever $\sum_{i=1}^{p} x_i \mathbf{v}_i = \mathbf{0}$, every x_i is zero.
- The columns of an $m \times n$ matrix A are linearly *independent* $\iff$ $\mathrm{A}\mathbf{x} = \mathbf{0}$ admits *only the trivial solution* $\iff$ there are n pivot positions (which requires $n \leq m$).
- A pair of vectors is linearly *dependent* $\iff$ they are collinear, i.e., one is a multiple of the other. Generally, linearly dependent sets admit linear dependence relations; some vector in the set is a linear combination of others.
- The columns of an $m \times n$ matrix are linearly *dependent* $\iff$ there exists a *nontrivial* solution to $\mathrm{A}\mathbf{x} = \mathbf{0}$ $\iff$ there is a free variable. $n > m \implies$ there is a free variable & the columns are dependent.
- If a collection of vectors contains the zero vector, then it is a linearly *dependent* set.