Describing Solution Sets to Linear Systems

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Outline

Previously...

We have seen that a linear system of m equations in n unknowns can be rephrased as a matrix-vector equation

$$\mathrm{A}\mathbf{x} = \mathbf{b}\,,$$

where A is the $m \times n$ real matrix of coefficients,

$$\mathbf{x} = \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix} \in \mathbb{R}^n$$

is the vector whose components are the n variables of the system, $\mathbf{b}$ is the column vector of constants, and $\mathrm{A}\mathbf{x}$ is the matrix-vector product, defined as the linear combination of the columns of A using $x_1, \ldots, x_n$ as the scalar weights.

Solution Sets

Now we seek to understand the *solution sets* of such equations: the hope is to be able to use the tools developed thus far to describe the set of all $\mathbf{x} \in \mathbb{R}^n$ satisfying a given equation $A\mathbf{x} = \mathbf{b}$.

To do this, we turn first to the easiest case to study: the case when $\mathbf{b} = \mathbf{0}$. Thus, we are asking about linear combinations of the column vectors of A which equal $\mathbf{0}$, or equivalently, intersections of linear subsets of $\mathbb{R}^n$ that all pass through the origin.

We will then discover that describing the solutions to $A\mathbf{x} = \mathbf{0}$ help unlock a general solution to $A\mathbf{x} = \mathbf{b}$ for any $\mathbf{b}$.

Homogeneous Systems

Definition

A system of m real linear equations in n variables is called *homogenous* if there exists an $m \times n$ matrix A such that the system can be described by the matrix-vector equation

$$A\mathbf{x} = \mathbf{0}\,,$$

where $\mathbf{x} \in \mathbb{R}^n$ is the vector whose components are the n variables of the system, and $\mathbf{0} \in \mathbb{R}^m$ is the zero vector with m components.

Observation

A homogeneous system is always consistent. In particular, it always has at least one (obvious) solution: the *trivial solution* $\mathbf{x} = \mathbf{0} \in \mathbb{R}^n$.

- The interesting question is thus whether, for a given matrix A, there exist nonzero vectors $\mathbf{x}$ satisfying $A\mathbf{x} = \mathbf{0}$.
- Such a solution $\mathbf{x}$ is called *nontrivial*.
- What must be true about A for $A\mathbf{x} = \mathbf{0}$ to have nontrivial solutions?
- For solutions to be non-unique, there must be at least one free variable, which implies there are fewer pivot positions than the number of variables.
- Thus $\mathrm{RREF}(A)$ has $< n$ leading 1s.

Handling Free Variables

- To solve $\mathrm{A}\mathbf{x} = \mathbf{0}$, one can perform the Gauss-Jordan reduction algorithm on $\left[\mathrm{A} \mid \mathbf{0}\right]$.
- If there are n pivot positions, then the solution is trivial (we don't even need to complete Gauss-Jordan!), so we are interested in the case when there are fewer than n pivot positions.
- Suppose that $k < n$ columns have no pivots. Then there are k free variables, and the the remaining $n - k$ variables can be expressed as linear combinations of the free ones.

Let us start with an example with a single free variable.

A 3×3 example

Example

Find the solutions to the homogeneous system $A\mathbf{x} = \mathbf{0}$ where $\mathbf{x}, \mathbf{0} \in \mathbb{R}^3$ and

$$A = \begin{bmatrix} 2 & 3 & -1 \\ 1 & 2 & 1 \\ 4 & 5 & -5 \end{bmatrix}.$$

Solution: To solve $A\mathbf{x} = \mathbf{0}$, we can row reduce the augmented matrix $\left[A \mid \mathbf{0}\right]$.

A 3×3 example

Example

After performing Gauss-Jordan, one obtains that

$$\left[\begin{array}{ccc|c} 2 & 3 & -1 & 0 \\ 1 & 2 & 1 & 0 \\ 4 & 5 & -5 & 0 \end{array}\right] \sim \left[\begin{array}{ccc|c} 1 & 0 & -5 & 0 \\ 0 & 1 & 3 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right],$$

which leaves the variable z free, and gives equations $x - 5z = 0$ and $y + 3z = 0$.
Thus,

$$\mathbf{x} = \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 5z \\ -3z \\ z \end{bmatrix} = z \begin{bmatrix} 5 \\ -3 \\ 1 \end{bmatrix}.$$

A 3×3 example

Example

Write $z = t$ and let

$$\mathbf{v} = \begin{bmatrix} 5 \\ -3 \\ 1 \end{bmatrix}.$$

We can thus describe the solution to this system as the set

$$\{t\mathbf{v} \mid t \in \mathbb{R}\} = \mathrm{Span}\{\mathbf{v}\}.$$

A 3×3 example

Example

When we write our solution explicitly in the form

$$\mathbf{x} = t\mathbf{v} = t\begin{bmatrix} 5 \\ -3 \\ 1 \end{bmatrix}, t \in \mathbb{R}$$

we say that the solution is in *parametric vector form*.

Parametric forms come in handy when one wants to tell a computer to draw the solution to a system.

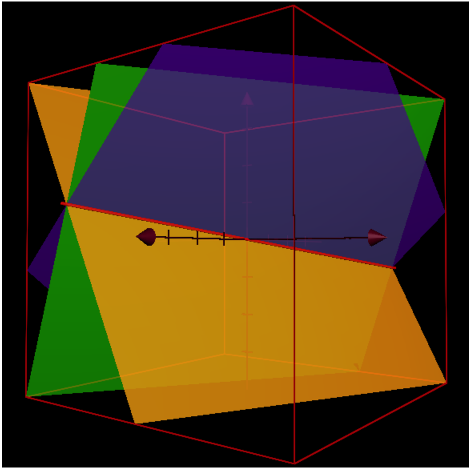

Figure: The three planes through the origin of $\mathbb{R}^3$, and their line of intersection

An Example with More Free Variables

Example

Suppose you have row reduced a 4×6 matrix A to obtain

$$\mathrm{RREF}(A) = \begin{bmatrix} 1 & 0 & 2 & 0 & -3 & 0 \\ 0 & 1 & -1 & 0 & 0 & 6 \\ 0 & 0 & 0 & 1 & 4 & -5 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}.$$

What can you say about nontrivial solutions to $A\mathbf{x} = \mathbf{0}$?

If we row reduced $\left[A \mid \mathbf{0}\right]$, we would obtain $\left[\mathrm{RREF}(A) \mid \mathbf{0}\right]$.

How Many Free Variables?

Example

Thus, all of the information necessary to solve the homogeneous system is contained in $\mathrm{RREF}(A)$. We just have to interpret it via equations.

Since there are 6 columns and only 3 pivot positions, we must have 3 free variables, coming from the three columns which are not pivot columns: x_3, x_5, and x_6.

As in the above example, we get equations allowing us to write x_1, x_2 and x_4 in terms of these free variables.

You've Won! 3 FREE Variables!

Example

Row one gives $x_1 = -2x_3 + 3x_5$, row two gives $x_2 = x_3 - 6x_6$, and row three gives $x_4 = -4x_5 + 5x_6$. Thus

$$\mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \\ x_6 \end{bmatrix} = \begin{bmatrix} -2x_3 + 3x_5 \\ x_3 - 6x_6 \\ x_3 \\ -4x_5 + 5x_6 \\ x_5 \\ x_6 \end{bmatrix} = x_3 \underbrace{\begin{bmatrix} -2 \\ 1 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}}_{\mathbf{u}} + x_5 \underbrace{\begin{bmatrix} 3 \\ 0 \\ 0 \\ -4 \\ 1 \\ 0 \end{bmatrix}}_{\mathbf{v}} + x_6 \underbrace{\begin{bmatrix} 0 \\ -6 \\ 0 \\ 5 \\ 0 \\ 1 \end{bmatrix}}_{\mathbf{w}}.$$

Thus, the solution set to $A\mathbf{x} = \mathbf{0}$ is $\mathrm{Span}\,\{\mathbf{u}, \mathbf{v}, \mathbf{w}\}$, or parametrically, $\mathbf{x} = r\mathbf{u} + s\mathbf{v} + t\mathbf{w}$ where $r, s, t \in \mathbb{R}$ are parameters.

Homogeneous Solution Sets

Definition

The solution set of a homogeneous equation $A\mathbf{x} = \mathbf{0}$ is called *the kernel of* A:

$$\ker A := \{\mathbf{x} \in \mathbb{R}^n \,|\, A\mathbf{x} = \mathbf{0}\}\,.$$

Note $\ker A \subset \mathbb{R}^n$. It is also called the *nullspace of* A.

We'll revisit kernels when we study linear transformations, for now I use it as a shorthand for the homogeneous solution set.

Kernels play an important role in the theory of linear transformations, and more generally, in the theory of certain algebraic maps called *homomorphisms*. Today we'll see that they play a role in describing non-unique solutions to general linear systems.

Inhomogeneous equations

We'll now begin to tackle the general case of $A\mathbf{x} = \mathbf{b}$ for nonzero $\mathbf{b}$, which is called the *inhomogeneous case*.

Before we prove the general result, let's look at a familiar example that contains all of the pieces.

Revisiting An Old Example

We had previously solved

$$\underbrace{\begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{bmatrix}}_{A} \underbrace{\begin{bmatrix} x \\ y \\ z \end{bmatrix}}_{\mathbf{x}} = \underbrace{\begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}}_{\mathbf{b}},$$

and discovered its solution was geometrically a line, given parametrically as

$$\mathbf{x} = \underbrace{\begin{bmatrix} 5/3 \\ -4/3 \\ 0 \end{bmatrix}}_{\mathbf{p}} + t \underbrace{\begin{bmatrix} 1 \\ -2 \\ 1 \end{bmatrix}}_{\mathbf{v}}, \quad t \in \mathbb{R}\,.$$

A Remark on Particular Solutions

Observe that taking $t = 0$, we find that $\mathbf{p}$ itself is a solution of the system: $\mathrm{A}\mathbf{p} = \mathbf{b}$. This is but one element in the solution set, and we'll call it a *particular solution* of $\mathrm{A}\mathbf{x} = \mathbf{b}$.

Remark

Note that $\mathbf{p}$ is not unique. We could, for example, take $t = 1$ in the above equation, and use vector addition to find a new particular solution $\mathbf{p}' = \mathbf{p} + \mathbf{v}$, and write the general solution as $\mathbf{x} = \mathbf{p}' + s\mathbf{v}, s \in \mathbb{R}$.

Geometrically, this just shifts the initial position on the line.

But what about the vector $\mathbf{v}$? What is $\mathrm{A}\mathbf{v}$?

$$\begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{bmatrix} \begin{bmatrix} 1 \\ -2 \\ 1 \end{bmatrix} = \begin{bmatrix} (1)(1) + (2)(-2) + (3)(1) \\ (4)(1) + (5)(-2) + (6)(1) \\ (7)(1) + (8)(-2) + (9)(1) \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}.$$

Thus $\mathbf{v} \in \ker \mathrm{A}$.

In fact, $\mathbf{v}$ spans the solution set to the homogeneous system $\mathrm{A}\mathbf{x} = \mathbf{0}$!

If we reexamine $\mathrm{RREF}(\mathrm{A})$ and follow the procedures for solving homogeneous systems, we recover $\mathbf{v}$ as a generator of $\ker \mathrm{A}$.

From Homogeneous to Inhomogeneous

For any system $\mathrm{A}\mathbf{x} = \mathbf{b}$, if we know a particular solution $\mathbf{p}$, how do we get a general solution?

Let $\mathbf{v} \in \ker \mathrm{A}$ be any solution to the homogeneous equation. Then observe

$$\mathrm{A}(\mathbf{p} + \mathbf{v}) = \mathrm{A}\mathbf{p} + \mathrm{A}\mathbf{v} = \mathbf{b} + \mathbf{0} = \mathbf{b}\,.$$

Thus, we obtain a new solution from any particular solution by adding elements of the kernel of A!

Is that everything?

But the question remains, can we write every solution to $\mathrm{A}\mathbf{x} = \mathbf{b}$ using some particular solution $\mathbf{p}$ and elements of the kernel of A?

If $\mathbf{p}'$ is any other solution of $\mathrm{A}\mathbf{x} = \mathbf{b}$, then

$$\mathrm{A}\mathbf{p} = \mathbf{b} = \mathrm{A}\mathbf{p}' \implies \mathrm{A}(\mathbf{p}' - \mathbf{p}) = \mathbf{b} - \mathbf{b} = \mathbf{0}\,.$$

Thus, the difference of any two particular solutions is a solution to the homogeneous system. That is, $\mathbf{p}' - \mathbf{p} \in \ker \mathrm{A}$!

Write $\mathbf{u} := \mathbf{p}' - \mathbf{p}$, and $\mathbf{p}' = \mathbf{p} + \mathbf{u}$ for $\mathbf{u} \in \ker \mathrm{A}$. Since $\mathbf{p}'$ is arbitrary, we've shown every solution can be written as a sum of some particular solution $\mathbf{p}$ and some $\mathbf{u} \in \ker \mathrm{A}$.

The General Solution

Theorem

Let $\mathbf{p} \in \mathbb{R}^n$ *be a vector such that* $A\mathbf{p} = \mathbf{b}$. *Then the solution set to the inhomogeneous equation* $A\mathbf{x} = \mathbf{b}$ *is*

$$\{\mathbf{p} + \mathbf{u} \,|\, \mathbf{u} \in \ker A\} \subset \mathbb{R}^n,$$

i.e., any solution $\mathbf{x}$ *of* $A\mathbf{x} = \mathbf{b}$ *can be expressed as the sum of the particular solution* $\mathbf{p}$ *and a solution* $\mathbf{u}$ *of the homogeneous system* $A\mathbf{u} = \mathbf{0}$.

Uniqueness versus Non-Uniqueness

Remark

Note that if the kernel of A is trivial, i.e., $\ker \mathrm{A} = \{\mathbf{0}\}$, then there must be a unique solution $\mathbf{p}$. Otherwise, if there is some other solution $\mathbf{p}' \neq \mathbf{p}$, our arguments above show $\mathbf{p}' - \mathbf{p} \in \ker \mathrm{A}$ is a *nontrivial* solution to the homogeneous equation, contradicting the triviality of $\ker \mathrm{A}$.

This gives us another theorem, essentially for free...

Uniqueness Theorem

Theorem

For an $m \times n$ linear system $\mathrm{A}\mathbf{x} = \mathbf{b}$, the following statements are logically equivalent:

1. *There exists a unique solution to the system,*
2. *The system is consistent and $\ker \mathrm{A}$ is trivial,*
3. *$n \leq m$ and $\left[\mathrm{A} \mid \mathbf{b}\right]$ has n pivot positions, all occurring within the first n columns.*

How to find and express the solution set

You are given a system $A\mathbf{x} = \mathbf{b}$.

1. Form the augmented matrix $\left[A \mid \mathbf{b}\right]$, and use row reduction to compute $\mathrm{RREF}\left(\left[A \mid \mathbf{b}\right]\right) = \left[\mathrm{RREF}(A) \mid \tilde{\mathbf{p}}\right]$.
2. The vector $\tilde{\mathbf{p}}$ helps you build a particular solution $\mathbf{p}$. Express each of the variables in terms of any free variables and entries of $\tilde{\mathbf{p}}$.
3. Write $\mathbf{x}$ as a vector in terms of any free variables using the previous step. Setting free variables all to 0 (or any other constant) gives a choice for $\mathbf{p}$.
4. Decompose $\mathbf{x}$ as a linear combination of vectors, including $\mathbf{p}$ and a collection of vectors each weighted by free variables.
5. This decomposition of $\mathbf{x}$ is a parametric vector form for the complete general solution to $A\mathbf{x} = \mathbf{b}$.

The vectors appearing with free variable/parameter weights generate the homogeneous solution set, i.e., they span the kernel ker A.

We'll later see that the pivot columns generate something else: *the image of the map* $\mathbf{x} \mapsto \mathrm{A}\mathbf{x}$ from $\mathbb{R}^n$ to $\mathbb{R}^m$. In particular, the *image* (also called the *range*), is all the points of $\mathbb{R}^m$ that can be written as linear combinations of the columns of A.

Thus, I'm claiming that only the pivot columns are needed in creating these linear combinations, building the image.

So there is a dichotomy: pivot columns contribute to the image, while non-pivot columns are associated to free variables, and the entries of non-pivot columns of $\mathrm{RREF}(\mathrm{A})$ help you build the kernel of A. We'll return to this.

The parametric equation of a line, revisited

Recall, a general parametric equation for a line is of the form $\mathbf{r}(t) = \mathbf{p} + t\mathbf{v}$, $t \in \mathbb{R}$.

We saw that when such a line arose as an intersection of planes in $\mathbb{R}^3$, the initial position $\mathbf{p}$ was a particular solution of the system of equations for the intersection of the planes, and the direction/"velocity" vector $\mathbf{v}$ was a generator of the solution set to the homogeneous system.

Geometrically, $\mathbf{v}$ is a vector that spans the line through the origin that one obtains by translating both planes so that they pass through the origin.

Another way to encounter a parameterization of a line is as a *normal line* to some linear subset.

Lines through $\mathbf{0} \in \mathbb{R}^2$ are kernels of dot products

Given a 2-vector

$$\mathbf{v} = \begin{bmatrix} v_1 \\ v_2 \end{bmatrix},$$

observe that we can easily find a vector $\mathbf{u}$ such that $\mathbf{u} \cdot \mathbf{v} = 0$. In particular take,

$$\pm\mathbf{u} = \begin{bmatrix} v_2 \\ -v_1 \end{bmatrix}.$$

Then $\mathbf{u} \cdot \mathbf{v} = u_1 v_1 + u_2 v_2 = \pm(v_2 v_1 - v_1 v_2) = 0$.

Thus, $\mathrm{Span}\,\{\mathbf{v}\} = \ker \begin{bmatrix} -v_2 & v_1 \end{bmatrix} = ker \begin{bmatrix} v_2 & -v_1 \end{bmatrix}$.

Slope-intercept form

The familiar slope-intercept form $y = mx + b$ thus describes a line by *writing down a linear system* whose solution is geometrically the line:

The system is $-mx + y = b$, with augmented matrix $\left[\begin{array}{cc|c} -m & 1 & b \end{array}\right]$. This has one pivot and one free variable.

$$\mathbf{p} = \begin{bmatrix} 0 \\ b \end{bmatrix}$$

is a particular solution, and

$$\mathbf{v} = \begin{bmatrix} 1 \\ m \end{bmatrix}$$

generates the homogeneous solution.

Slope-intercept Form

Thus, **v** is the kernel of the dot product with any vector perpendicular to the line, such as

$$\begin{bmatrix} -m \\ 1 \end{bmatrix}.$$

We conclude that the line $y = mx + b$ has a parametric description

$$\mathbf{x} = \mathbf{p} + t\mathbf{v} = \begin{bmatrix} 0 \\ b \end{bmatrix} + t \begin{bmatrix} 1 \\ m \end{bmatrix} = \begin{bmatrix} t \\ b + tm \end{bmatrix}.$$

The point-slope form is also easily parameterized. The point-slope equation of a line

$$y - y_0 = m(x - x_0)\,,$$

corresponds to a parameterization

$$\mathbf{x} = \mathbf{p}' + t\mathbf{v} = \begin{bmatrix} x_0 \\ y_0 \end{bmatrix} + t \begin{bmatrix} 1 \\ m \end{bmatrix} = \begin{bmatrix} x_0 + t \\ y_0 + tm \end{bmatrix}\,,$$

which is the same solution set, described using a different particular solution!

Planes as a system

Similarly, we can describe planes in the language of systems, by regarding a plane equation

$$ax + by + cz = d$$

as a simple system

$$\begin{bmatrix} a & b & c \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = [d] .$$

Let us look at the form of a general solution. Pick some initial position $\mathbf{x}_0$ known to satisfy the plane equation. This is a particular solution. Thus, $\mathbf{n} \cdot \mathbf{x}_0 = d$. Let $\mathbf{x} \neq \mathbf{x}_0$ be another point on the plane.

From particular to general

Since $\mathbf{x}$ is a solution, $\mathbf{x} - \mathbf{x}_0$ must be a solution of

$$ax + by + cz = 0\,.$$

Indeed: computing
$a(x - x_0) + b(y - y_0) + c(z - z_0) = ax + by + cz - d = 0.$
Write

$$\mathbf{n} = \begin{bmatrix} a \\ b \\ c \end{bmatrix}.$$

Planes and Normals

Our calculation above gives that $\mathbf{n} \cdot (\mathbf{x} - \mathbf{x}_0) = 0$!

View $\mathbf{x} - \mathbf{x}_0$ as a *displacement vector* along the plane.

Appealing to the fact that $\mathbf{n} \neq \mathbf{0}$ and $\mathbf{x} - \mathbf{x}_0 \neq \mathbf{0}$, and $\mathbf{n} \cdot (\mathbf{x} - \mathbf{x}_0) = \|\mathbf{n}\| \|\mathbf{x} - \mathbf{x}_0\| \cos\theta$ where θ is the angle between the vectors $\mathbf{n}$ and $\mathbf{x} - \mathbf{x}_0$, we have demonstrated that $\theta = \pi/2 + k\pi$, i.e., $\mathbf{n} \perp \mathbf{x} - \mathbf{x}_0$.

Thus, we see that $\mathbf{n}$ is a vector *perpendicular* to the plane. It is called a *normal vector*.

Span of two vectors and parameterization

We know we can also describe a plane through $\mathbf{0} \in \mathbb{R}^3$ as the span of two vectors. These vectors are generate the kernel of the dot product map $\mathbf{x} \mapsto \mathbf{n} \cdot \mathbf{x}$ for some normal vector $\mathbf{n}$ to the plane.

We can see this explicitly by rewriting the plane equation. Assume $a \neq 0$. Then

$$\mathrm{RREF}\left(\left[\begin{array}{ccc|c} a & b & c & 0 \end{array}\right] \right) = \left[\begin{array}{ccc|c} 1 & b/a & c/a & 0 \end{array}\right] .$$

Thus, we have a solution

$$\mathbf{x} = \begin{bmatrix} -(b/a)y - (c/a)z \\ y \\ z \end{bmatrix} = y \begin{bmatrix} -b/a \\ 1 \\ 0 \end{bmatrix} + z \begin{bmatrix} -c/a \\ 0 \\ 1 \end{bmatrix} .$$

Parameterizing a plane

If $a = 0$ then we instead have x as a free variable, and can solve for either y or z (one of which must have a nonzero coefficient for the plane equation to be meaningful).

We can adapt this idea to parameterize any plane given an equation for it. Consider the general plane

$$ax + by + cz = d\,.$$

Parameterizing a plane

If $a \neq 0$, then

$$\mathbf{x} = \begin{bmatrix} d/a \\ 0 \\ 0 \end{bmatrix} + y\begin{bmatrix} -b/a \\ 1 \\ 0 \end{bmatrix} + z\begin{bmatrix} -c/a \\ 0 \\ 1 \end{bmatrix}.$$

If we take $s = -ay$ and $t = -az$, then we can rewrite this as

$$\mathbf{x} = \begin{bmatrix} d/a \\ 0 \\ 0 \end{bmatrix} + s\begin{bmatrix} b \\ -a \\ 0 \end{bmatrix} + t\begin{bmatrix} c \\ 0 \\ -a \end{bmatrix}.$$

Normal Lines

Finally, we can also describe a line as the set normal to some plane. If we know the equation of the plane, we can easily read off a normal. But what if we only know two vectors spanning the plane, and some initial point?

This is a challenge problem (building on one proposed in the slides from lecture 5): given arbitrary vectors $\mathbf{u}$ and $\mathbf{v}$ spanning a plane through a point d, find the equation of the plane, and use the ideas from the preceding discussion to express a line through $\mathbf{0}$ normal to the plane as the solution of some homogeneous system, in terms of the components of $\mathbf{u}$ and $\mathbf{v}$. Then describe the normal line to this plane through a given point $\mathbf{x}_0 \in \mathbb{R}^3$ parametrically. (If you know what a cross-product is, don't use them in your solution. In a sense, you are re-deriving the idea of a cross product, up to scaling, through linear algebra.)

Homework for Week 3

- Next week we cover sections 1.7 and 1.8 of the textbook. Please try to read through up to 1.8 by Friday, 2/9.
- Tuesday 2/6 homework for sections 1.3 and 1.4 are due in MyMathLab.
- Thursday 2/8 section 1.5 is due in MyMathLab.
- Quiz 1 is due Monday, 2/5 at the beginning of class.