Review:

- $A, B \in \mathbb{R}^{n\times n}$ similar matrices if there's an invertible $P \in \mathbb{R}^{n\times n}$: $A = PBP^{-1}$.
- Similar matrices have the same characteristic polynomial; they share eigenvalues, and any shared eigenvalue has the same multiplicities (both algebraic & geometric) for A & B.
- We also saw that geometric multiplicity is $\leq$ algebraic multiplicity. In particular, if $A = PBP^{-1}$, and $\vec{x}$ is an eigenvector with eigenvalue λ, for A, then $P^{-1}\vec{x}$ is an eigenvector for B with eigenvalue λ.

Using eigentheory, we will develop criteria to determine when a matrix A is similar to a diagonal matrix D, i.e. we'll describe a method to either produce a factorization $A = PDP^{-1}$, or demonstrate that none exists.

Diagonal matrices and Exponentiation

If $D = \begin{bmatrix} d_1 & 0 & \cdots & 0 \\ 0 & d_2 & \cdots & 0 \\ \vdots & & \ddots & \\ 0 & & & d_n \end{bmatrix} =: \mathrm{diag}(d_1, \dots d_n)$

then for any positive integer k,

$$D^k = \begin{bmatrix} d_1^k & 0 & \cdots & 0 \\ 0 & d_2^k & \cdots & 0 \\ \vdots & & \ddots & \\ 0 & & & d_n^k \end{bmatrix} = \mathrm{diag}(d_1^k, \dots, d_n^k).$$

Example: Let $A = \begin{bmatrix} 7 & 2 \\ -4 & 1 \end{bmatrix}$. Given that $A = PDP^{-1}$ for $P = \begin{bmatrix} 1 & 1 \\ -1 & -2 \end{bmatrix}$ and $D = \begin{bmatrix} 5 & 0 \\ 0 & 3 \end{bmatrix}$, Find a formula for A^k, $k \in \mathbb{Z}$.

Solution: Observe, if $A = PDP^{-1}$, then
$$A^2 = PDP^{-1}PDP^{-1} = PD^2P^{-1}.$$

More generally

$$A^k = PD^kP^{-1} \text{ for } k \text{ a positive integer.}$$

What if k is a negative integer?

Observe $A^{-1} = (PDP^{-1})^{-1} = (P^{-1})^{-1}D^{-1}(P)^{-1}$
$= PD^{-1}P^{-1}$.

If $D = \mathrm{diag}(d_1, \dots d_n)$, then
$$D^{-1} = \mathrm{diag}(d_1^{-1}, \dots, d_n^{-1}).$$

So, for $A = \begin{bmatrix} 1 & 1 \\ -1 & -2 \end{bmatrix}\begin{bmatrix} 5 & 0 \\ 0 & 3 \end{bmatrix}\begin{bmatrix} 1 & 1 \\ -1 & -2 \end{bmatrix}^{-1}$,

$$A^k = \begin{bmatrix} 1 & 1 \\ -1 & -2 \end{bmatrix}\begin{bmatrix} 5^k & 0 \\ 0 & 3^k \end{bmatrix}\frac{1}{1(-2)-(-1)(1)}\begin{bmatrix} -2 & -1 \\ 1 & 1 \end{bmatrix}$$
$$= \begin{bmatrix} 1 & 1 \\ -1 & -2 \end{bmatrix}\begin{bmatrix} 5^k & 0 \\ 0 & 3^k \end{bmatrix}\begin{bmatrix} 2 & 1 \\ -1 & -1 \end{bmatrix}$$

Thus $A^k = \begin{bmatrix} 7 & 2 \\ -4 & 1 \end{bmatrix}^k = \begin{bmatrix} 2(5^k) - 3^k & 5^k - 3^k \\ 2(3^k - 5^k) & 2(3^k) - 5^k \end{bmatrix}$

Definition: A square matrix A is said to be <u>diagonalizable</u> if it is similar to a diagonal matrix, i.e., $A = PDP^{-1}$ for an invertible matrix P and a diagonal matrix D.

The Diagonalization Theorem

$A \in \mathbb{R}^{n\times n}$ is (real) diagonalizable if and only if A has n linearly independent eigenvectors in $\mathbb{R}^n$.

Moreover, if $A = PDP^{-1}$ for a diagonal matrix D, then the columns of P are n linearly independent eigenvectors of A, and the diagonal entries of D are the eigenvalues of A, that correspond, respectively, to the eigenvectors in P.

Proof: Let $D = \operatorname{diag}(\lambda_1, \dots \lambda_n)$ for some real numbers (not necessarily distinct) $\lambda_1, \dots, \lambda_n \in \mathbb{R}$, and suppose A is diagonalizable with $A = PDP^{-1}$. Write $P = [\vec{v}_1 \cdots \vec{v}_n]$.

Observe: $PD = [\vec{v}_1 \cdots v_n]\begin{bmatrix} \lambda_1 & 0 & \cdots & 0 \\ 0 & \lambda_2 & \cdots & 0 \\ \vdots & \vdots & & \vdots \\ 0 & 0 & \cdots & \lambda_n \end{bmatrix}$

$= [\lambda_1\vec{v}_1 \ \lambda_2\vec{v}_2 \cdots \lambda_n\vec{v}_n]$.

Since $A = PDP^{-1}$, $AP = PD$

$\Rightarrow [A\vec{v}_1 \cdots A\vec{v}_n] = [\lambda_1\vec{v}_1 \cdots \lambda_n\vec{v}_n]$.

$\Rightarrow A\vec{v}_i = \lambda_i \vec{v}_i$ for $i = 1, \dots n$.

Since P is invertible, $\{\vec{v}_1 \cdots \vec{v}_n\}$ is a linearly independent set, and thus $\vec{v}_i \neq \vec{0}$ for any i. Thus the $\vec{v}_i$'s are eigenvectors of A with eigenvalues $\lambda_1 \cdots \lambda_n$ respectively.

Conversely, if A has n linearly indep. eigenvectors $\vec{v}_1, \dots, \vec{v}_n$, then

$\underbrace{A[\vec{v}_1 \cdots \vec{v}_n]}_{P} = [\lambda_1\vec{v}_1 \cdots \lambda_n\vec{v}_n]$

$= P \underbrace{\operatorname{diag}(\lambda_1 \cdots \lambda_n)}_{D}$

for eigenvalues $\lambda_1, \dots \lambda_n$. Thus A is diagonalizable, as P is invertible and $A = PDP^{-1}$. ∎

Example: Diagonalize the following matrix, if possible:

$$A = \begin{bmatrix} 1 & 3 & 3 \\ -3 & -5 & -3 \\ 3 & 3 & 1 \end{bmatrix}$$

Thus, we seek invertible P, whose columns are eigenvectors of A, and a diagonal matrix whose diagonal entries are eigenvalues of A.

1.) To find eigenvalues, we compute

$\chi_A(\lambda) = \det(A - \lambda I_3) = -\lambda^3 - 3\lambda^2 + 4$.

This factors nicely:

$\chi_A(\lambda) = -(\lambda-1)(\lambda+2)^2$.

Eigenvalues are roots $\lambda = 1$ & $\lambda = -2$ of $\chi_A(\lambda) = 0$. Note $\lambda = -2$ has alg. multiplicity 2.

2.) We need to find 3 linearly independent eigenvectors. For $\lambda = 1$:

$$[A - \lambda I_3 \mid 0] = \left[\begin{array}{ccc|c} 0 & 3 & 3 & 0 \\ -3 & -6 & -3 & 0 \\ 3 & 3 & 0 & 0 \end{array}\right] \sim \left[\begin{array}{ccc|c} 1 & 0 & -1 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

$\vec{v}_1 = \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix}$ is an eigenvector for $\lambda = 1$.

$\lambda = -2$:

$$[A + 2I_3 \mid 0] = \left[\begin{array}{ccc|c} 3 & 3 & 3 & 0 \\ -3 & -3 & -3 & 0 \\ 3 & 3 & 3 & 0 \end{array}\right] \sim \left[\begin{array}{ccc|c} 1 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

$\vec{v}_2 = \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix}$ $\vec{v}_3 = \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix}$ are eigenvectors for $\lambda = -2$.

3.) $P = [\vec{v}_1 \ \vec{v}_2 \ \vec{v}_3] = \begin{bmatrix} 1 & 1 & 1 \\ -1 & -1 & 0 \\ 1 & 0 & -1 \end{bmatrix}$

4.) $D = \begin{bmatrix} 1 & 0 & 0 \\ 0 & -2 & 0 \\ 0 & 0 & -2 \end{bmatrix}$.

It's good to check:

$$AP = \begin{bmatrix} 1 & 3 & 3 \\ -3 & -5 & -3 \\ 3 & 3 & 1 \end{bmatrix} \begin{bmatrix} 1 & 1 & 1 \\ -1 & -1 & 0 \\ 1 & 0 & -1 \end{bmatrix} = \begin{bmatrix} 1 & -2 & -2 \\ -1 & 2 & 0 \\ 1 & 0 & 2 \end{bmatrix}$$

$$PD = \begin{bmatrix} 1 & 1 & 1 \\ -1 & -1 & 0 \\ 1 & 0 & -1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & -2 & 0 \\ 0 & 0 & -2 \end{bmatrix} = \begin{bmatrix} 1 & -2 & -2 \\ -1 & 2 & 0 \\ 1 & 0 & 2 \end{bmatrix}.$$

Note that in our example,

- the eigenvalue -2 with algebraic multiplicity 2 also had geometric multiplicity 2. By the theorem on similarity, it is necessary that geometric and algebraic multiplicities be equal, and that multiplicities sum to n for a matrix to be real diagonalizable.

Thm: If $A \in \mathbb{R}^{n\times n}$ has n distinct real eigenvalues, then A is diagonalizable.

If A has repeated eigenvalues, then A is diagonalizable if and only if the geometric multiplicities of all of its real eigenvalues sum to n, which happens if and only if two conditions are met:

i) $\chi_A(\lambda)$ factors completely as a product of real linear factors

ii) For each λ_i an eigenvalue, the algebraic and geometric multiplicities are equal:

$$m(\lambda_i) = \mu(\lambda_i) := \dim E_{\lambda_i}$$

where $m(\lambda_i)$ is the highest degree of a linear factor $(\lambda - \lambda_i)$ dividing $\chi_A(\lambda)$, and $\mu(\lambda_i)$ is the dimension of the λ_i-eigenspace E_{λ_i}.

Example: Diagonalize the matrix, if possible:

$$A = \begin{bmatrix} 5 & 0 & 0 & 0 \\ 0 & 5 & 0 & 0 \\ 1 & 4 & -3 & 0 \\ -1 & -2 & 0 & -3 \end{bmatrix}$$

Exercise (let them work on it; they may discuss with eachother).

$$P = \begin{bmatrix} -8 & -16 & 0 & 0 \\ 4 & 4 & 0 & 0 \\ 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \end{bmatrix}$$

$$D = \begin{bmatrix} 5 & 0 & 0 & 0 \\ 0 & 5 & 0 & 0 \\ 0 & 0 & -3 & 0 \\ 0 & 0 & 0 & -3 \end{bmatrix}.$$

Extra Questions:

- Are Diagonalizations unique when they exist?
- If A is Diagonalizable, what about A^T?
- Can a Diagonalizable matrix be non-invertible? If yes, give an explicit 2×2 example. Otherwise, prove your claim.