Determinants of Square Matrices

Computing (Hyper-)Volumes and Testing Invertibility

A. Havens

Department of Mathematics
University of Massachusetts, Amherst

February 28- March 3, 2018

Outline

The 2×2 Determinant

Recall that for a 2×2 matrix

$$\mathrm{A} = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}$$

the quantity $\Delta = a_{11}a_{22} - a_{12}a_{21}$ determines if A is invertible:

$$\mathrm{A}^{-1} \text{ exists} \iff \Delta \neq 0\,.$$

Definition

The quantity $\det(\mathrm{A}) := \Delta(\mathrm{A}) = a_{11}a_{22} - a_{12}a_{21}$ is called the *determinant* of the 2×2 matrix A.

For the following discussion we consider an arbitrary real 2×2 matrix A with determinant $\det \mathrm{A} = \Delta$.

The 2×2 Determinant as Area

Geometrically we can interpret the meaning of Δ as follows. If $A = \begin{bmatrix} \mathbf{a}_1 & \mathbf{a}_2 \end{bmatrix}$ then Δ is the *signed area* of the parallelogram with vertices whose positions are given by $\mathbf{0}$, $\mathbf{a}_1$, $\mathbf{a}_2$, and $\mathbf{a}_1 + \mathbf{a}_2$.

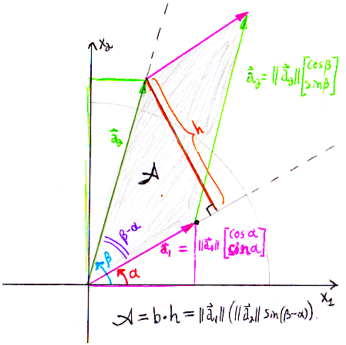
We call this the *parallelogram spanned by the two vectors* $\mathbf{a}_1$ *and* $\mathbf{a}_2$*.*

The sign will be determined by the spatial relationship of $\mathbf{a}_1$ to $\mathbf{a}_2$. We will explain this below via a notion of *orientations*.

We can prove the relation to area using elementary trigonometry.

Some Elementary Geometry

Assume for the moment that $0 \leq \alpha < \beta \leq \pi/2$ so the vectors lie in the first quadrant, and consider the diagram.



Some Trigonometry

The length of the base of the parallelogram can be taken to be $b = \|\mathbf{a}_1\|$, and the length of the altitude is $h = \|\mathbf{a}_2\| \sin(\beta - \alpha)$.

The area $\mathcal{A}$ of the parallelogram is related to the determinant Δ via the sine angle subtraction identity:

Some Trigonometry

The length of the base of the parallelogram can be taken to be $b = \|\mathbf{a}_1\|$, and the length of the altitude is $h = \|\mathbf{a}_2\| \sin(\beta - \alpha)$.

The area $\mathcal{A}$ of the parallelogram is related to the determinant Δ via the sine angle subtraction identity:

$$\mathcal{A} = b \cdot h$$

Some Trigonometry

The length of the base of the parallelogram can be taken to be $b = \|\mathbf{a}_1\|$, and the length of the altitude is $h = \|\mathbf{a}_2\| \sin(\beta - \alpha)$.

The area $\mathcal{A}$ of the parallelogram is related to the determinant Δ via the sine angle subtraction identity:

$$\mathcal{A} = b \cdot h = \|\mathbf{a}_1\| \|\mathbf{a}_2\| \sin(\beta - \alpha)$$

Some Trigonometry

The length of the base of the parallelogram can be taken to be $b = \|\mathbf{a}_1\|$, and the length of the altitude is $h = \|\mathbf{a}_2\| \sin(\beta - \alpha)$.

The area $\mathcal{A}$ of the parallelogram is related to the determinant Δ via the sine angle subtraction identity:

$$\begin{aligned}\mathcal{A} = b \cdot h &= \|\mathbf{a}_1\|\|\mathbf{a}_2\| \sin(\beta - \alpha)\\ &= \|\mathbf{a}_1\|\|\mathbf{a}_2\|\Big(\sin(\beta)\cos(\alpha) - \cos(\beta)\sin(\alpha)\Big)\end{aligned}$$

Some Trigonometry

The length of the base of the parallelogram can be taken to be $b = \|\mathbf{a}_1\|$, and the length of the altitude is $h = \|\mathbf{a}_2\| \sin(\beta - \alpha)$.

The area $\mathcal{A}$ of the parallelogram is related to the determinant Δ via the sine angle subtraction identity:

$$\begin{aligned}\mathcal{A} = b \cdot h &= \|\mathbf{a}_1\|\|\mathbf{a}_2\| \sin(\beta - \alpha)\\ &= \|\mathbf{a}_1\|\|\mathbf{a}_2\| \big(\sin(\beta)\cos(\alpha) - \cos(\beta)\sin(\alpha)\big)\\ &= \|\mathbf{a}_1\|\|\mathbf{a}_2\| \sin(\beta)\cos(\alpha) - \|\mathbf{a}_1\|\|\mathbf{a}_2\| \cos(\beta)\sin(\alpha)\end{aligned}$$

Some Trigonometry

The length of the base of the parallelogram can be taken to be $b = \|\mathbf{a}_1\|$, and the length of the altitude is $h = \|\mathbf{a}_2\| \sin(\beta - \alpha)$.

The area $\mathcal{A}$ of the parallelogram is related to the determinant Δ via the sine angle subtraction identity:

$$\begin{aligned}
\mathcal{A} = b \cdot h &= \|\mathbf{a}_1\|\|\mathbf{a}_2\| \sin(\beta - \alpha) \\
&= \|\mathbf{a}_1\|\|\mathbf{a}_2\| \big(\sin(\beta)\cos(\alpha) - \cos(\beta)\sin(\alpha)\big) \\
&= \|\mathbf{a}_1\|\|\mathbf{a}_2\| \sin(\beta)\cos(\alpha) - \|\mathbf{a}_1\|\|\mathbf{a}_2\| \cos(\beta)\sin(\alpha) \\
&= \big(\|\mathbf{a}_1\| \sin(\beta)\big)\big(\|\mathbf{a}_2\| \cos(\alpha)\big) - \big(\|\mathbf{a}_2\| \cos(\beta)\big)\big(\|\mathbf{a}_1\| \sin(\alpha)\big)
\end{aligned}$$

Some Trigonometry

The length of the base of the parallelogram can be taken to be $b = \|\mathbf{a}_1\|$, and the length of the altitude is $h = \|\mathbf{a}_2\| \sin(\beta - \alpha)$.

The area $\mathcal{A}$ of the parallelogram is related to the determinant Δ via the sine angle subtraction identity:

$$\begin{aligned}
\mathcal{A} = b \cdot h &= \|\mathbf{a}_1\| \|\mathbf{a}_2\| \sin(\beta - \alpha) \\
&= \|\mathbf{a}_1\| \|\mathbf{a}_2\| \Big(\sin(\beta)\cos(\alpha) - \cos(\beta)\sin(\alpha) \Big) \\
&= \|\mathbf{a}_1\| \|\mathbf{a}_2\| \sin(\beta)\cos(\alpha) - \|\mathbf{a}_1\| \|\mathbf{a}_2\| \cos(\beta)\sin(\alpha) \\
&= \Big(\|\mathbf{a}_1\| \sin(\beta) \Big) \Big(\|\mathbf{a}_2\| \cos(\alpha) \Big) - \Big(\|\mathbf{a}_2\| \cos(\beta) \Big) \Big(\|\mathbf{a}_1\| \sin(\alpha) \Big) \\
&= a_{11}a_{22} - a_{12}a_{21} = \Delta \,.
\end{aligned}$$

Some Trigonometry

The length of the base of the parallelogram can be taken to be $b = \|\mathbf{a}_1\|$, and the length of the altitude is $h = \|\mathbf{a}_2\| \sin(\beta - \alpha)$.

The area $\mathcal{A}$ of the parallelogram is related to the determinant Δ via the sine angle subtraction identity:

$$\begin{aligned}
\mathcal{A} = b \cdot h &= \|\mathbf{a}_1\| \|\mathbf{a}_2\| \sin(\beta - \alpha) \\
&= \|\mathbf{a}_1\| \|\mathbf{a}_2\| \big(\sin(\beta)\cos(\alpha) - \cos(\beta)\sin(\alpha) \big) \\
&= \|\mathbf{a}_1\| \|\mathbf{a}_2\| \sin(\beta)\cos(\alpha) - \|\mathbf{a}_1\| \|\mathbf{a}_2\| \cos(\beta)\sin(\alpha) \\
&= \big(\|\mathbf{a}_1\| \sin(\beta) \big) \big(\|\mathbf{a}_2\| \cos(\alpha) \big) - \big(\|\mathbf{a}_2\| \cos(\beta) \big) \big(\|\mathbf{a}_1\| \sin(\alpha) \big) \\
&= a_{11}a_{22} - a_{12}a_{21} = \Delta\,.
\end{aligned}$$

It is left as an exercise to check that one still gets $\pm\Delta$ as the area if the vectors are not in the first quadrant. When will the determinant be the negative of the area?

Orientations in the Plane

Note that if instead $0 \leq \beta < \alpha \leq \pi/2$, the sign is reversed (as $\sin(\beta - \alpha) < 0$), whence, if $\mathbf{a}_1$ would rotate *clockwise* through the parallelogram towards $\mathbf{a}_2$ instead of counterclockwise, Δ would return the *negative* of the area.

This corresponds to the algebraic fact that if we swap the columns of A the determinant changes sign:

$$\det\left(\begin{bmatrix} \mathbf{a}_2 & \mathbf{a}_1 \end{bmatrix}\right) = -\det\left(\begin{bmatrix} \mathbf{a}_1 & \mathbf{a}_2 \end{bmatrix}\right).$$

Only one configuration gives the positive area: the configuration *oriented* in the same way as the (ordered) standard basis $(\mathbf{e}_1, \mathbf{e}_2)$.

Orientations in the Plane

Definition

The pair $(\mathbf{a}_1, \mathbf{a}_2)$ is *positively oriented* if and only if $\mathbf{a}_1$ lies to the right of $\mathbf{a}_2$ as sides of the parallelogram they span, otherwise it is negatively oriented.

Intuitively, the determinant of a 2×2 matrix A is positive if the matrix A maps $(\mathbf{e}_1, \mathbf{e}_2)$ to $(\mathbf{a}_1, \mathbf{a}_2)$ in such a way that the unit square is mapped to the parallelogram spanned by $\mathbf{a}_1$ and $\mathbf{a}_2$ while preserving the "sense" of the angle sweeping from $\mathbf{e}_1$ to $\mathbf{e}_2$ at the vertex $\mathbf{0}$.

Namely, this angle is counterclockwise for $(\mathbf{e}_1, \mathbf{e}_2)$, and so should be counterclockwise for $(\mathbf{a}_1, \mathbf{a}_2)$ for Δ to be the area. Else, it will be the negative of the area.

Orientations in Other Dimensions

For dimension 3 we'll need to replace this notion with *right-handedness* of the columns of A.

For dimensions $n > 3$, we'll actually first define determinants, and then use the sign of the determinant to define orientations.

The intuition will be that positive determinants correspond to linear transformations that can be accomplished without reflections, while negative determinants occur when there is a decomposition of the map involving an odd number of reflections.

A formula for the 3×3 determinant

Let $\mathrm{A} = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$, let A_{ij} denote the 2×2 submatrix obtained from A by deleting the ith row and jth column, and let $|\mathrm{B}| := \det(\mathrm{B})$ for any 2×2 matrix B. Then

Definition

The determinant of a 3×3 matrix A as above is the quantity

$$\begin{aligned} \det \mathrm{A} &:= a_{11} \det \mathrm{A}_{11} - a_{12}\mathrm{A}_{12} + a_{13}\mathrm{A}_{13} \\ &= a_{11} \begin{vmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{vmatrix} - a_{12} \begin{vmatrix} a_{21} & a_{23} \\ a_{31} & a_{33} \end{vmatrix} + a_{13} \begin{vmatrix} a_{21} & a_{22} \\ a_{31} & a_{32} \end{vmatrix} \end{aligned}$$

Another formula for the 3×3 determinant

There's also a rule which somewhat generalizes the 2×2 determinant rule:

$$\begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix} = a_{11}a_{22}a_{33} + a_{12}a_{23}a_{31} + a_{13}a_{21}a_{32}$$

$$- a_{31}a_{22}a_{13} - a_{32}a_{23}a_{11} - a_{33}a_{21}a_{12} \,.$$

This is called the "Scheme of Sarrus." You should check that this formula agrees with the above formula.

Testing Invertibility

Using A as above, one can show by row reduction that, if $a_{11} \neq 0$ then there is a row equivalence

$$\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix} \sim \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ 0 & \det \mathrm{A}_{33} & \det \mathrm{A}_{32} \\ 0 & 0 & a_{11} \det \mathrm{A} \end{bmatrix},$$

whence the if the matrix A is invertible, it must be true that $\det \mathrm{A} \neq 0$. What happens if $a_{11} = 0$? One can also prove the converse: if $\det \mathrm{A} \neq 0$, then A is invertible (Try it!)

Cross Products

There is a closely related formula for a *vector product*, called the cross-product. We procure here some notation and terminology to define it geometrically.

For $\mathbf{u}, \mathbf{v} \in \mathbb{R}^3$ not parallel, let $\mathscr{P}$ be the parallelogram whose vertices have positions $\mathbf{0}$, $\mathbf{u}$, $\mathbf{v}$ and $\mathbf{u} + \mathbf{v}$, and let $\mathcal{A}(\mathscr{P})$ be the area of this parallelogram. You should convince yourself (draw a picture!) that this area is equal to $\|\mathbf{u}\|\|\mathbf{v}\| \sin\theta$ where $\theta \in (0, \pi)$ is the angle of separation of $\mathbf{u}$ and $\mathbf{v}$.

A triple $(\mathbf{a}_1, \mathbf{a}_2, \mathbf{a}_3)$ of linearly independent 3-vectors is called *right-handed* if $\det \begin{bmatrix} \mathbf{a}_1 & \mathbf{a}_2 & \mathbf{a}_3 \end{bmatrix} > 0$. One can prove that there is a unique unit vector $\hat{\mathbf{n}}_{\mathscr{P}}$ normal to the parallelogram $\mathscr{P}$ (and so, perpendicular to both $\mathbf{u}$ and $\mathbf{v}$) such that $(\mathbf{u}, \mathbf{v}, \hat{\mathbf{n}}_{\mathscr{P}})$ is a right-handed triple.

Defining Cross-Products Geometrically

Definition

For $\mathbf{u}, \mathbf{v} \in \mathbb{R}^3$, the cross product is the vector defined by the following conditions:

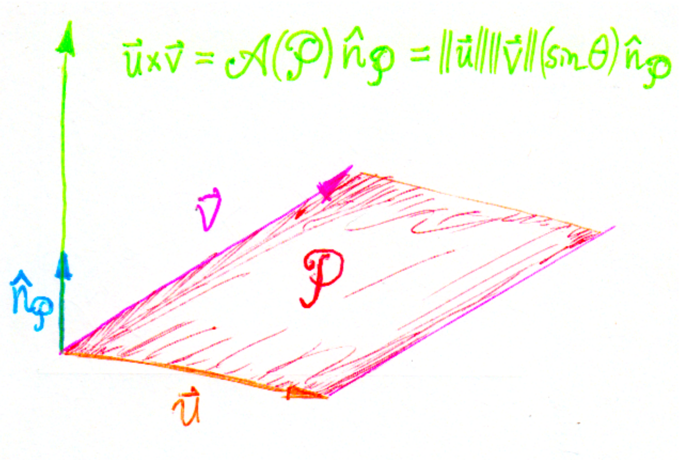
i. $\mathbf{u} \times \mathbf{v} = \mathbf{0}$ whenever $\mathbf{u}$ and $\mathbf{v}$ are parallel.

ii. For $\mathbf{u}$ and $\mathbf{v}$ linearly independent, with $\mathscr{P}$, $\hat{\mathbf{n}}_{\mathscr{P}}$, $\mathcal{A}(\mathscr{P})$, and θ as above,

$$\mathbf{u} \times \mathbf{v} := \mathcal{A}(\mathscr{P})\hat{\mathbf{n}}_{\mathscr{P}} = \|\mathbf{u}\|\|\mathbf{v}\| \sin(\theta)\,\hat{\mathbf{n}}_{\mathscr{P}}\,.$$

Cross products do not generalize to all dimensions (there exist generalizations in 7 dimensions, and one can consider the usual product of real numbers as a kind of “trivial” cross product of 1-vectors).

Visualization of a Cross Product



A Cross Product Formula

Proposition

The three dimensional cross product is the 3-vector valued product given by

$$(\mathbf{u}, \mathbf{v}) \mapsto \mathbf{u} \times \mathbf{v} = \begin{vmatrix} u_2 & u_3 \\ v_2 & v_3 \end{vmatrix} \mathbf{e}_1 - \begin{vmatrix} u_1 & u_3 \\ v_1 & v_3 \end{vmatrix} \mathbf{e}_2 + \begin{vmatrix} u_1 & u_2 \\ v_1 & v_2 \end{vmatrix} \mathbf{e}_3 \,,$$

where $\mathbf{u} = u_1\mathbf{e}_1 + u_2\mathbf{e}_2 + u_3\mathbf{e}_3$ *and* $\mathbf{v} = v_1\mathbf{e}_1 + v_2\mathbf{e}_2 + v_3\mathbf{e}_3$.

It is instructive to try to prove this from the above geometric definition.

Triple Products and Volume

Definition

The *Vector Triple Product* of three vectors $\mathbf{u}, \mathbf{v}, \mathbf{w} \in \mathbb{R}^3$ is the determinant

$$\begin{vmatrix} \mathbf{u} & \mathbf{v} & \mathbf{w} \end{vmatrix} = \mathbf{u} \cdot (\mathbf{v} \times \mathbf{w}) .$$

Claim

The vector triple product of $\mathbf{u}, \mathbf{v}, \mathbf{w} \in \mathbb{R}^3$ computes the volume of the *parallelepiped*
$\{x_1\mathbf{u} + x_2\mathbf{v} + x_3\mathbf{w} \,|\, 0 \leq x_1, x_2, x_3 \leq 1\} \subset \mathbb{R}^3$.

Challenge Problem: Show the equality of the determinant $\begin{vmatrix} \mathbf{u} & \mathbf{v} & \mathbf{w} \end{vmatrix}$ and the expression $\mathbf{u} \cdot (\mathbf{v} \times \mathbf{w})$, and then prove the above claim.

Some Properties

One should check the following properties of cross and triple products.

- $(s\mathbf{u})\times\mathbf{v} = s(\mathbf{u}\times\mathbf{v}) = \mathbf{u}\times(s\mathbf{v})$,
- $\mathbf{u}\times\mathbf{v} = -(\mathbf{v}\times\mathbf{u})$,
- $\mathbf{u}\times\mathbf{u} = \mathbf{0}$,
- $\mathbf{u}\times(\mathbf{v}+\mathbf{w}) = \mathbf{u}\times\mathbf{v}+\mathbf{u}\times\mathbf{w}$,
- $\mathbf{u}\cdot(\mathbf{v}\times\mathbf{w}) = \mathbf{w}\cdot(\mathbf{u}\times\mathbf{v})$,
- $\mathbf{u}\cdot(\mathbf{v}\times\mathbf{w}) = -\mathbf{v}\cdot(\mathbf{u}\times\mathbf{w})$.

Challenge Problem: Find a matrix representing the linear map $\mathbf{u}\times : \mathbb{R}^3 \to \mathbb{R}^3$ that maps $\mathbf{v}$ to $\mathbf{u}\times\mathbf{v}$.

Multiple Vector Inputs

To define determinants for $n \times n$ matrices, we briefly will consider functions taking multiple vectors as inputs, and giving a scalar output.

We've defined 2×2 and 3×3 determinants as numbers associated to these square matrices, computed from their entries. Now, we will consider inputting the columns of a matrix into some function, and having it spit out a number which we'd like to have certain properties (like measuring if the columns are linearly independent, computing a signed volume...)

We can then define the determinant of a matrix A to be the output of a suitable choice of such function applied to the columns of A. So what properties must such a function have?

Multilinearity

Definition

A map $T : \mathbb{R}^n \times \mathbb{R}^n \times \ldots \times \mathbb{R}^n \to \mathbb{R}$ is called *multilinear* if it is linear in each variable, i.e., if for any scalars $\alpha, \beta \in \mathbb{R}$, and vectors $\mathbf{x}_1, \ldots, \mathbf{x}_i, \ldots \in \mathbb{R}^n$ and $\mathbf{y}_i \in \mathbb{R}^n$:

$$T(\mathbf{x}_1, \ldots, \alpha\mathbf{x}_i + \beta\mathbf{y}_i, \ldots) = \alpha T(\mathbf{x}_1, \ldots, \mathbf{x}_i, \ldots) + \beta T(\mathbf{x}_1, \ldots, \mathbf{y}_i, \ldots) .$$

You should check that for a 2×2 or 3×3 matrix A the map taking the columns as inputs and outputting the determinant $\det \mathrm{A}$ as defined above is multilinear.

Bilinearity

A special case of multilinearity we've already seen is *bilinearity*.

E.g. the map $\cdot : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}$ which takes a pair of vectors $(\mathbf{u}, \mathbf{v})$ to their dot product $\mathbf{u} \cdot \mathbf{v}$ is bilinear, since

$$\begin{aligned}(\alpha\mathbf{u} + \beta\mathbf{w}) \cdot (\gamma\mathbf{v} + \delta\mathbf{y}) &= \alpha\mathbf{u} \cdot (\gamma\mathbf{v} + \delta\mathbf{y}) + \beta\mathbf{w} \cdot (\gamma\mathbf{v} + \delta\mathbf{y}) \\ &= \alpha\gamma\mathbf{u} \cdot \mathbf{v} + \alpha\delta\mathbf{u} \cdot \mathbf{y} + \beta\gamma\mathbf{w} \cdot \mathbf{v} + \beta\delta\mathbf{w} \cdot \mathbf{y}\,.\end{aligned}$$

Observe however that $\mathbf{u} \cdot \mathbf{v} = \mathbf{v} \cdot \mathbf{u}$. We want a map where swapping two arguments changes the sign of the output.

For vectors $\mathbf{a}_1, \mathbf{a}_2 \in \mathbb{R}^2$, the map $(\mathbf{a}_1, \mathbf{a}_2) \mapsto \det\left(\begin{bmatrix} \mathbf{a}_1 & \mathbf{a}_2 \end{bmatrix}\right)$ is bilinear, but also changes sign when we swap $\mathbf{a}_1$ and $\mathbf{a}_2$.

Alternating Multilinear Maps

Definition

A multilinear map $T : \mathbb{R}^n \times \mathbb{R}^n \times \ldots \times \mathbb{R}^n \to \mathbb{R}$ is called *alternating* if for any pair of indices $i < j$

$$T(\mathbf{x}_1, \ldots, \mathbf{x}_i, \ldots \mathbf{x}_j, \ldots) = -T(\mathbf{x}_1, \ldots, \mathbf{x}_j, \ldots \mathbf{x}_i, \ldots),$$

i.e., swapping any pair of distinct inputs scales the value by -1.

Observation

$T(\mathbf{x}_1, \ldots, \mathbf{x}_i, \ldots x_j, \ldots) = 0$ if $\mathbf{x}_i = \mathbf{x}_j$ and T is alternating.
Indeed, if a real number t satisfies $t = -t$, then t must equal zero.

Being an alternating multilinear map is quite restrictive. Any repeated inputs force the map to be zero, and any linear combinations appearing in the inputs allow the value to be re-expressed as a linear combination of images summed over various input sets.

In particular, we will see that we can understand alternating multilinear maps

$$T : \underbrace{\mathbb{R}^n \times \ldots \times \mathbb{R}^n}_{n \text{ times}} \to \mathbb{R}$$

by examining the value of T on the standard basis (as an ordered n-tuple of n-vectors).

A Lemma

Lemma

An alternating multilinear map

$$T : \underbrace{\mathbb{R}^n \times \ldots \times \mathbb{R}^n}_{n \text{ times}} \to \mathbb{R}$$

is determined uniquely by its value $T(\mathbf{e}_1, \ldots, \mathbf{e}_n)$*, where* $(\mathbf{e}_1, \ldots, \mathbf{e}_n)$ *is the standard, ordered basis of* $\mathbb{R}^n$*.*

In the language of vector spaces that we will be developing soon, we would say that "the space of maps from $\mathbb{R}^{n\times n}$ to $\mathbb{R}$, that are alternating and multilinear as maps on the columns, is a one dimensional real vector space. Thus, up to rescaling, there is a unique nontrivial alternating multilinear map from $\mathbb{R}^{n\times n}$ to $\mathbb{R}$."

Arguing the Lemma

We need to use both multilinearity and alternativity of the map.

To do so without too many gory summations, we'll talk briefly about an idea which is very important throughout mathematics (and quite central to determinants, even if elementary treatments go to great pains to avoid or gloss over them): *permutations*.

Definition

A permutation of the symbols $\{1, 2, \ldots, n\}$ is a bijective (one-to-one and onto) map $\sigma : \{1, 2, \ldots, n\} \to \{1, 2, \ldots, n\}$.

For example $(1, 2, 3, 4, 5, 6) \mapsto (3, 4, 1, 6, 5, 2)$ is a permutation.

Why Do We Need Permutations?

Assume $T : \underbrace{\mathbb{R}^n \times \ldots \times \mathbb{R}^n}_{n \text{ times}} \to \mathbb{R}$ is a multilinear alternating map.

Our lemma is like an analogue of the theorem that a linear map is determined by its effect on the standard basis.

Recall, for a linear map M we get a unique matrix representative:

$$M(\mathbf{x}) = \mathrm{A}\mathbf{x} = \left[\begin{array}{ccc} M(\mathbf{e}_1) & \ldots & M(\mathbf{e}_n) \end{array}\right] \mathbf{x}\,.$$

For our multilinear map, we have multiple inputs, and there are many, many ways to select inputs chosen from the standard basis for large n.

Exponential Difficulties

In fact, there are n^n ways to choose n inputs from the n basis elements, with repetition allowed.

For a general multilinear map T with n inputs from $\mathbb{R}^n$, we could determine T from the values of $T(\mathbf{e}_{i_1}, \ldots, \mathbf{e}_{i_j}, \ldots, \mathbf{e}_{i_n})$, for all n^n different indicial combinations $i_1, \ldots, i_j, \ldots, i_n \in \{1, \ldots n\}$.

This would follow from expressing any input $\mathbf{x}_i$ in the standard basis and applying multilinearity, until $T(\mathbf{x}_1, \ldots, \mathbf{x}_n)$ is expressed as a linear combination of terms $T(\mathbf{e}_{i_1}, \ldots, \mathbf{e}_{i_n})$.

Alternation to the Rescue

But since we assume T is also alternating, we have far fewer terms to consider.

Indeed, $T(\mathbf{e}_1, \ldots, \mathbf{e}_i, \ldots, \mathbf{e}_i, \ldots \mathbf{e}_n) = 0$. So we only have to consider the images of T applied to the standard basis, in some order, i.e., we will examine how the value of T is determined by the values of $T(\mathbf{e}_{\sigma(1)}, \ldots, \mathbf{e}_{\sigma(i)}, \ldots, \mathbf{e}_{\sigma(n)})$, where σ is any permutation of $\{1, 2, \ldots, n\}$.

There are only $n!$ permutations, which is quite a bit smaller than n^n for large n.

But the lemma claims more strongly that it will suffice to know just one value to determine T: we only need to know the value of $T(\mathbf{e}_1, \ldots, \mathbf{e}_n)$.

Even and Odd Permutations

Permutations come in two flavors: *even* and *odd*.

The idea is that any rearrangement of the symbols $1, \ldots, n$ can be accomplished by some sequence of swaps $i \leftrightarrow j$, called *transpositions*.

There may be many sequences of transpositions resulting in the same rearrangement $(1, 2, \ldots, n) \mapsto \big(\sigma(1), \sigma(2), \ldots, \sigma(n)\big)$, but one can show that for a given σ it will either require an even or an odd number of transpositions.

If σ requires an even number of transpositions, it is called an *even* permutation, otherwise it is called odd.

The Sign of a Permutation

We define the *sign* of a permutation to be $+1$ if the permutation is even, and -1 if the permutation is odd. The *trivial permutation*, which changes nothing, is even. Why?

Performing a transposition twice leaves the order of the symbols unchanged, hence the trivial permutation "factors" in various ways as a composition of even sequences of transpositions.

We write $\mathrm{sgn}(\sigma)$ for the sign of a permutation σ.

For example, the permutation $(1, 2, 3, 4, 5, 6) \overset{\sigma}{\mapsto} (3, 4, 1, 6, 5, 2)$ is odd so $\mathrm{sgn}(\sigma) = -1$ in this case. Indeed, σ requires just 3 transpositions: first $1 \leftrightarrow 3$, then $2 \leftrightarrow 4$, then $2 \leftrightarrow 6$.

Alternating Maps and Permutations

Let us return to considering an alternating multilinear map $T : \underbrace{\mathbb{R}^n \times \ldots \times \mathbb{R}^n}_{n \text{ times}} \to \mathbb{R}$.

Since any transposition of the arguments of an alternating map changes the sign, it follows that for any alternating map and any permutation σ of the indices $1, \ldots, n$

$$T(\mathbf{x}_{\sigma(1)}, \ldots, \mathbf{x}_{\sigma(i)}, \ldots, \mathbf{x}_{\sigma(n)}) = \operatorname{sgn}(\sigma) T(\mathbf{x}_1, \ldots, \mathbf{x}_i, \ldots, \mathbf{x}_n) .$$

In particular, $T(\mathbf{e}_{\sigma(1)}, \ldots, \mathbf{e}_{\sigma(i)}, \ldots, \mathbf{e}_{\sigma(n)}) = \operatorname{sgn}(\sigma) T(\mathbf{e}_1, \ldots, \mathbf{e}_n)$.

Multilinear Combinations

Writing $\mathbf{x}_j = x_{1j}\mathbf{e}_1 + \ldots + x_{nj}\mathbf{e}_n = \sum_{i=1}^n x_{ij}\mathbf{e}_i$ and appealing to multilinearity, we have

$$T(\mathbf{x}_1, \ldots, \mathbf{x}_i, \ldots, \mathbf{x}_n) = T\left(\mathbf{x}_1, \ldots, \sum_{i=1}^n x_{ij}\mathbf{e}_i, \ldots, \mathbf{x}_n\right)$$
$$= \sum_{i=1}^n x_{ij}\, T(\mathbf{x}_1, \ldots, \mathbf{e}_i, \ldots, \mathbf{x}_n)\,.$$

Performing a similar expansion for each input, one gets a linear combination of a great many terms of the form $T(\mathbf{e}_{i_1}, \ldots, \mathbf{e}_{i_n})$.

And Finally. . . a formula!

But by alternativity, $T(\mathbf{e}_{i_1}, \dots, \mathbf{e}_{i_n})$ can be nonzero only if the tuple $(\mathbf{e}_{i_1}, \dots, \mathbf{e}_{i_n})$ is of the form $(\mathbf{e}_{\sigma(1)}, \dots, \mathbf{e}_{\sigma(n)})$ for some permutation σ, in which case it equals $\mathrm{sgn}\,(\sigma)\, T(\mathbf{e}_1, \dots, \mathbf{e}_n)$.

It follows that:

$$T(\mathbf{x}_1, \dots, \mathbf{x}_n) = \left(\sum_{\sigma \in \mathfrak{S}_n} \mathrm{sgn}\,(\sigma) \prod_{i=1}^{n} x_{i\sigma(i)} \right) T(\mathbf{e}_1, \dots, \mathbf{e}_n)\,,$$

where $\mathfrak{S}_n$ is the *group* of all permutations of $\{1, 2, \dots, n\}$.

The Theorem

Theorem

There exists a unique alternating multilinear map $D : \underbrace{\mathbb{R}^n \times \ldots \times \mathbb{R}^n}_{n \text{ times}} \to \mathbb{R}$ *such that*

i. $D(\mathbf{e}_1, \ldots, \mathbf{e}_n) = 1$, *where* $(\mathbf{e}_1, \ldots, \mathbf{e}_n)$ *is the standard basis,*

ii. *If* $\mathrm{A} = \begin{bmatrix} \mathbf{a}_1 & \ldots & \mathbf{a}_n \end{bmatrix}$ *and* $\mathrm{B} = \begin{bmatrix} \mathbf{b}_1 & \ldots & \mathbf{b}_n \end{bmatrix}$, *then*

$$\begin{aligned} D(\mathrm{A}\mathbf{b}_1, \ldots, \mathrm{A}\mathbf{b}_n) &= D(\mathbf{a}_1, \ldots, \mathbf{a}_n) D(\mathbf{b}_1, \ldots, \mathbf{b}_n) \\ &= D(\mathrm{B}\mathbf{a}_1, \ldots, \mathrm{B}\mathbf{a}_n)\,. \end{aligned}$$

The Definition

Definition

Given a square matrix $A = \begin{bmatrix} \mathbf{a}_1 & \dots & \mathbf{a}_n \end{bmatrix} \in \mathbb{R}^{n\times n}$, the *determinant of* A is the quantity

$$\det(A) = D(\mathbf{a}_1, \dots, \mathbf{a}_n)$$

where D is the map described by the preceding theorem.

Testing it out on the 2×2 case

Let $\mathrm{A} = \begin{bmatrix} \mathbf{a}_1 & \mathbf{a}_2 \end{bmatrix} = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}$.

Write $\mathbf{a}_1 = a_{11}\mathbf{e}_1 + a_{21}\mathbf{e}_2$ and $\mathbf{a}_2 = a_{12}\mathbf{e}_1 + a_{22}\mathbf{e}_2$.

We will verify that $\det(\mathrm{A}) = a_{11}a_{22} - a_{21}a_{12}$ using the definition.

By the definition, $\det(\mathrm{A}) = D(\mathbf{a}_1, \mathbf{a}_2)$, where the map $D : \mathbb{R}^2 \times \mathbb{R}^2 \to \mathbb{R}$ is the unique bilinear, alternating map determined by $D(\mathbf{e}_1, \mathbf{e}_2) = 1$.

Testing it out on the 2×2 case

By alternativity, $D(\mathbf{e}_1, \mathbf{e}_2) = 1 \implies D(\mathbf{e}_2, \mathbf{e}_1) = -1$, and $D(\mathbf{e}_1, \mathbf{e}_1) = 0 = D(\mathbf{e}_2, \mathbf{e}_2)$. Applying these facts together with multilinearity:

Testing it out on the 2×2 case

By alternativity, $D(\mathbf{e}_1, \mathbf{e}_2) = 1 \implies D(\mathbf{e}_2, \mathbf{e}_1) = -1$, and $D(\mathbf{e}_1, \mathbf{e}_1) = 0 = D(\mathbf{e}_2, \mathbf{e}_2)$. Applying these facts together with multilinearity:

$$\det(\mathrm{A}) = D(\mathbf{a}_1, \mathbf{a}_2) = D(a_{11}\mathbf{e}_1 + a_{21}\mathbf{e}_2, a_{12}\mathbf{e}_1 + a_{22}\mathbf{e}_2)$$

Testing it out on the 2×2 case

By alternativity, $D(\mathbf{e}_1, \mathbf{e}_2) = 1 \implies D(\mathbf{e}_2, \mathbf{e}_1) = -1$, and $D(\mathbf{e}_1, \mathbf{e}_1) = 0 = D(\mathbf{e}_2, \mathbf{e}_2)$. Applying these facts together with multilinearity:

$$\begin{aligned}\det(\mathrm{A}) = D(\mathbf{a}_1, \mathbf{a}_2) &= D(a_{11}\mathbf{e}_1 + a_{21}\mathbf{e}_2, a_{12}\mathbf{e}_1 + a_{22}\mathbf{e}_2) \\ &= a_{11}D(\mathbf{e}_1, a_{12}\mathbf{e}_1 + a_{22}\mathbf{e}_2) + a_{21}D(\mathbf{e}_2, a_{12}\mathbf{e}_1 + a_{22}\mathbf{e}_2)\end{aligned}$$

Testing it out on the 2×2 case

By alternativity, $D(\mathbf{e}_1, \mathbf{e}_2) = 1 \implies D(\mathbf{e}_2, \mathbf{e}_1) = -1$, and $D(\mathbf{e}_1, \mathbf{e}_1) = 0 = D(\mathbf{e}_2, \mathbf{e}_2)$. Applying these facts together with multilinearity:

$$\begin{aligned}\det(\mathrm{A}) &= D(\mathbf{a}_1, \mathbf{a}_2) = D(a_{11}\mathbf{e}_1 + a_{21}\mathbf{e}_2, a_{12}\mathbf{e}_1 + a_{22}\mathbf{e}_2)\\ &= a_{11}D(\mathbf{e}_1, a_{12}\mathbf{e}_1 + a_{22}\mathbf{e}_2) + a_{21}D(\mathbf{e}_2, a_{12}\mathbf{e}_1 + a_{22}\mathbf{e}_2)\\ &= a_{11}a_{12}D(\mathbf{e}_1, \mathbf{e}_1) + a_{11}a_{22}D(\mathbf{e}_1, \mathbf{e}_2)\\ &\qquad + a_{21}a_{12}D(\mathbf{e}_2, \mathbf{e}_1) + a_{21}a_{22}D(\mathbf{e}_2, \mathbf{e}_2)\end{aligned}$$

Testing it out on the 2×2 case

By alternativity, $D(\mathbf{e}_1, \mathbf{e}_2) = 1 \implies D(\mathbf{e}_2, \mathbf{e}_1) = -1$, and $D(\mathbf{e}_1, \mathbf{e}_1) = 0 = D(\mathbf{e}_2, \mathbf{e}_2)$. Applying these facts together with multilinearity:

$$\begin{aligned}
\det(\mathrm{A}) &= D(\mathbf{a}_1, \mathbf{a}_2) = D(a_{11}\mathbf{e}_1 + a_{21}\mathbf{e}_2, a_{12}\mathbf{e}_1 + a_{22}\mathbf{e}_2) \\
&= a_{11}D(\mathbf{e}_1, a_{12}\mathbf{e}_1 + a_{22}\mathbf{e}_2) + a_{21}D(\mathbf{e}_2, a_{12}\mathbf{e}_1 + a_{22}\mathbf{e}_2) \\
&= a_{11}a_{12}D(\mathbf{e}_1, \mathbf{e}_1) + a_{11}a_{22}D(\mathbf{e}_1, \mathbf{e}_2) \\
&\qquad + a_{21}a_{12}D(\mathbf{e}_2, \mathbf{e}_1) + a_{21}a_{22}D(\mathbf{e}_2, \mathbf{e}_2) \\
&= a_{11}a_{22}D(\mathbf{e}_1, \mathbf{e}_2) + a_{21}a_{12}D(\mathbf{e}_2, \mathbf{e}_1)
\end{aligned}$$

Testing it out on the 2×2 case

By alternativity, $D(\mathbf{e}_1, \mathbf{e}_2) = 1 \implies D(\mathbf{e}_2, \mathbf{e}_1) = -1$, and $D(\mathbf{e}_1, \mathbf{e}_1) = 0 = D(\mathbf{e}_2, \mathbf{e}_2)$. Applying these facts together with multilinearity:

$$\begin{aligned}
\det(\mathrm{A}) = D(\mathbf{a}_1, \mathbf{a}_2) &= D(a_{11}\mathbf{e}_1 + a_{21}\mathbf{e}_2, a_{12}\mathbf{e}_1 + a_{22}\mathbf{e}_2) \\
&= a_{11}D(\mathbf{e}_1, a_{12}\mathbf{e}_1 + a_{22}\mathbf{e}_2) + a_{21}D(\mathbf{e}_2, a_{12}\mathbf{e}_1 + a_{22}\mathbf{e}_2) \\
&= a_{11}a_{12}D(\mathbf{e}_1, \mathbf{e}_1) + a_{11}a_{22}D(\mathbf{e}_1, \mathbf{e}_2) \\
&\qquad + a_{21}a_{12}D(\mathbf{e}_2, \mathbf{e}_1) + a_{21}a_{22}D(\mathbf{e}_2, \mathbf{e}_2) \\
&= a_{11}a_{22}D(\mathbf{e}_1, \mathbf{e}_2) + a_{21}a_{12}D(\mathbf{e}_2, \mathbf{e}_1) \\
&= a_{11}a_{22} - a_{21}a_{12}\,.
\end{aligned}$$

The Difficult Formula of Leibniz

From our work above in discussing the lemma, we could have opted to instead define the determinant by a direct formula involving the entries of the matrix A.

Many texts define the determinant in such a way; we'll discuss a common, recursive definition below.

The appropriate formula from our discussion, for a matrix A with entries (a_{ij}), is

$$\det(\mathrm{A}) = \sum_{\sigma \in \mathfrak{S}_n} \mathrm{sgn}\,(\sigma) \prod_{i=1}^{n} a_{i\sigma(i)}\,.$$

This is called the *Leibniz formula for the determinant of* A.

Oh, Leibniz. . .

To prove the existence statement of the theorem above, it would suffice to check that this function satisfies the conditions of the theorem (i.e., that it is multilinear and alternating in the columns, evaluates to 1 for the identity matrix, and multiplicative for the matrix product.)

But how useful is this formula for computing?
The Leibniz formula is not just a bit of a mystery to stare at, it's also not a nice way to compute! Try using it to verify the formula for a 3×3 determinant, and you will understand the desire for a cleaner definition and means of computation.

Can we not. . .

Even the recursive definition we'll discuss below (which ultimately comes from this formula) is not an efficient way to compute determinants for large n (though it has advantages if there are many entries equal to 0 present)

But from our definition, we've ensured desirable properties for the determinant, which will later allow us to use a row-reduction procedure to help us compute determinants.

We now list and discuss the basic properties of determinants which follow from its definition.

Elementary Properties of the Determinant

Proposition

The following properties of the determinant hold for any n:

1. $\det \mathrm{I}_n = 1$,
2. $\det(\mathrm{AB}) = \det(\mathrm{A})\det(\mathrm{B}) = \det(\mathrm{BA})$ *for any square matrices* $\mathrm{A}, \mathrm{B} \in \mathbb{R}^{n\times n}$,
3. *If* $\mathrm{A} = \begin{bmatrix} \mathbf{a}_1 & \dots & \mathbf{a}_n \end{bmatrix} \in \mathbb{R}^{n\times n}$, *and* $s \in \mathbb{R}$ *is any scalar, then*
   - $\det\left(\begin{bmatrix} \mathbf{a}_1 & \dots & s\mathbf{a}_j & \dots & \mathbf{a}_n \end{bmatrix}\right) = s\det(\mathrm{A})$
   - $\det\left(\begin{bmatrix} \mathbf{a}_1 & \dots & \mathbf{a}_j + s\mathbf{a}_k & \dots, & \mathbf{a}_n \end{bmatrix}\right) = \det(\mathrm{A})$,
   - $\det\left(\begin{bmatrix} \mathbf{a}_1 & \dots & \mathbf{a}_i & \dots & \mathbf{a}_j & \dots & \mathbf{a}_n \end{bmatrix}\right)$
     $= -\det\left(\begin{bmatrix} \mathbf{a}_1 & \dots & \mathbf{a}_j & \dots & \mathbf{a}_i & \dots & \mathbf{a}_n \end{bmatrix}\right)$,
4. $\det(\mathrm{A}) = \det(\mathrm{A}^{\mathrm{t}})$.

Row Operations and the Determinant

Since $\det(\mathrm{A}) = \det(\mathrm{A}^{\mathrm{t}})$, it follows from the properties describing column manipulation that we can track how a determinant changes under row operations. In particular:

- Swapping two columns changes the sign of the determinant, and a row swap $R_i \leftrightarrow R_j$ also changes the sign of the determinant. More generally, if you permute either rows or columns by a permutation σ, the determinant is multiplied by $\mathrm{sgn}(\sigma)$.
- If you scale a column by a scalar $s \in \mathbb{R}$, the determinant scales by s as well. Thus the row operation $sR_i \mapsto R_i$ scales the determinant by s.
- The elementary row replacement $R_i + sR_j \mapsto R_i$ preserves the determinant, since the corresponding column operation does.

Linear Dependence and the Determinant

Combining the previous two facts, the operation

$$sR_i + \sum_{1\leq j\neq i\leq n} t_j R_j \mapsto R_i$$

scales the determinant by s.

A consequence of this is that if the rows are linearly dependent, one can use a dependence relation to zero out a row R_i (with the s in the above operation chosen to be 1), but this is then the same as scaling a row by 0, and so the determinant becomes zero. But the row operation above doesn't change the determinant if $s = 1$, so the original determinant must be zero.

Thus, if the rows (equivalently, the columns) are linearly dependent, then the determinant is zero.

Invertible Matrices and Determinants

One can thus add a statement about determinants to the invertible matrix theorem.

A matrix A is invertible if and only if $\det \mathrm{A} \neq 0$, which is then equivalent to the many other characterizations of invertible matrices: The columns must span $\mathbb{R}^n$, the transformation is injective, there are pivots in every row and column, and so $\mathrm{RREF}(\mathrm{A}) = \mathrm{I}_n$, et cetera.

Also note that the multiplicativity of determinants implies that, for A invertible,

$$\det(\mathrm{A}) = \frac{1}{\det(\mathrm{A}^{-1})}\,.$$

Cofactors

Recall, for a 3×3 matrix, we could express the determinant using determinants of 2×2 submatrices scaled by entries from some row of A (or some column), and given appropriate signs.

We will generalize this idea to a formula for determinants of $n \times n$ matrices, called the *cofactor expansion* (also called *the Laplace expansion*). First, we define cofactors.

Definition

Let A be an $n \times n$ matrix and let A_{ij} denote the $(n-1) \times (n-1)$ submatrix of A obtained by deleting the ith row and jth column of A. Then the (i,j)th-cofactor of A is the determinant $C_{ij} = \det \mathrm{A}_{ij}$.

The Cofactor Expansion of Laplace

Proposition

Let A *be an* $n \times n$ *matrix with entries* (a_{ij}). *Fix an index* $i \in \{1, \ldots, n\}$. *Then*

$$\det(\mathrm{A}) = \sum_{j=1}^{n} (-1)^{i+j} a_{ij} C_{ij} \,,$$

where $C_{ij} = \det \mathrm{A}_{ij}$ *is the* (i,j)*th-cofactor of* A.

This is called the *cofactor expansion along the* i*th row*. Equivalently one can form the cofactor expansion along the jth column for some fixed j:

$$\det(\mathrm{A}) = \sum_{i=1}^{n} (-1)^{i+j} a_{ij} C_{ij} \,.$$

Proof? Not Today...

To prove the cofactor expansion, it would suffice to check that any cofactor expansion is an alternating and multi-linear map with respect to the columns of A, and that it equals 1 for the identity matrix. This is left as an exercise.

Another approach, which shows *where it comes from*, is to extract it from the Leibniz formula. This is incredibly tedious, so I won't subject you to it (you can find a version of a derivation on wikipedia, if you must.)

Instead, let's take as a given that this formula checks out, and examine and use the cofactor expansion to get a feel for computing determinants.

An Example

Example

Use cofactor expansion along the middle column to verify the equation for a 3×3 determinant.

Let $\mathrm{A} = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$. For any matrix B denote by $|\mathrm{B}|$ the determinant $\det(\mathrm{B})$ (this notation is a common shorthand.) E.g., one may write

$$\det(\mathrm{A}) = \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix}.$$

Example: Verifying the 3×3 Determinant Formula

$$\begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix} = (-1)^{1+2} a_{12} C_{12} + (-1)^{2+2} a_{22} C_{22} + (-1)^{3+2} a_{32} C_{32}$$

Example: Verifying the 3×3 Determinant Formula

$$\begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix} = (-1)^{1+2} a_{12} C_{12} + (-1)^{2+2} a_{22} C_{22} + (-1)^{3+2} a_{32} C_{32}$$

$$= -a_{12} \begin{vmatrix} a_{21} & a_{23} \\ a_{31} & a_{33} \end{vmatrix} + a_{22} \begin{vmatrix} a_{11} & a_{13} \\ a_{31} & a_{33} \end{vmatrix} - a_{32} \begin{vmatrix} a_{11} & a_{13} \\ a_{21} & a_{23} \end{vmatrix}$$

Example: Verifying the 3×3 Determinant Formula

$$\begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix} = (-1)^{1+2} a_{12} C_{12} + (-1)^{2+2} a_{22} C_{22} + (-1)^{3+2} a_{32} C_{32}$$

$$= -a_{12} \begin{vmatrix} a_{21} & a_{23} \\ a_{31} & a_{33} \end{vmatrix} + a_{22} \begin{vmatrix} a_{11} & a_{13} \\ a_{31} & a_{33} \end{vmatrix} - a_{32} \begin{vmatrix} a_{11} & a_{13} \\ a_{21} & a_{23} \end{vmatrix}$$

$$= -a_{12}(a_{21}a_{33} - a_{31}a_{23}) + a_{22}(a_{11}a_{33} - a_{31}a_{13}) - a_{32}(a_{11}a_{23} - a_{21}a_{13})$$

Example: Verifying the 3×3 Determinant Formula

$$\begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix} = (-1)^{1+2} a_{12} C_{12} + (-1)^{2+2} a_{22} C_{22} + (-1)^{3+2} a_{32} C_{32}$$

$$= -a_{12} \begin{vmatrix} a_{21} & a_{23} \\ a_{31} & a_{33} \end{vmatrix} + a_{22} \begin{vmatrix} a_{11} & a_{13} \\ a_{31} & a_{33} \end{vmatrix} - a_{32} \begin{vmatrix} a_{11} & a_{13} \\ a_{21} & a_{23} \end{vmatrix}$$

$$= -a_{12}(a_{21}a_{33} - a_{31}a_{23}) + a_{22}(a_{11}a_{33} - a_{31}a_{13}) - a_{32}(a_{11}a_{23} - a_{21}a_{13})$$

$$= a_{11}a_{22}a_{33} + a_{12}a_{23}a_{31} + a_{13}a_{21}a_{32} - a_{13}a_{22}a_{33} - a_{11}a_{23}a_{32} - a_{12}a_{21}a_{33}\,.$$

The Pattern of Signs

Note that the sign term $(-1)^{i+j}$ implies that the cofactor expansion accounts for evenness/oddness of permutations by assigning signs to the positions of the matrix according to a "checkerboard" pattern :

$$\begin{bmatrix} + & - & + & \dots \\ - & + & - & \dots \\ + & - & + & \dots \\ \vdots & \vdots & \vdots & \ddots \end{bmatrix}$$

Recursive Computation

The cofactor expansion allows the determinant to be computed recursively; however note that to compute, e.g., a determinant for a general 4×4 matrix, one must compute 4 separate 3×3 determinants, each of which requires 3 separate 2×2 determinant computations.

In general, the complexity of the calculation grows factorially in the size of the matrix.

But the cofactor expansion performs well for by-hand calculation of determinants for small matrices populated with numerous 0 entries.

An Example

Example

Use a cofactor expansion to compute the determinant of the matrix

$$A = \begin{bmatrix} 3 & 1 & 0 & 5 \\ 0 & 4 & -1 & 0 \\ 7 & 1 & 0 & 6 \\ -8 & 0 & 2 & 0 \end{bmatrix}.$$

Example: Cofactor Expansion with Zeros

$$\begin{vmatrix} 3 & 1 & 0 & 5 \\ 0 & 4 & -1 & 0 \\ 7 & 1 & 0 & 6 \\ -8 & 0 & 2 & 0 \end{vmatrix}$$

Example: Cofactor Expansion with Zeros

$$\begin{vmatrix} 3 & 1 & 0 & 5 \\ 0 & 4 & -1 & 0 \\ 7 & 1 & 0 & 6 \\ -8 & 0 & 2 & 0 \end{vmatrix} = 0 - (-1)\begin{vmatrix} 3 & 1 & 5 \\ 7 & 1 & 6 \\ -8 & 0 & 0 \end{vmatrix} + 0 - 2\begin{vmatrix} 3 & 1 & 5 \\ 0 & 4 & 0 \\ 7 & 1 & 6 \end{vmatrix}$$

Example: Cofactor Expansion with Zeros

$$\begin{vmatrix} 3 & 1 & 0 & 5 \\ 0 & 4 & -1 & 0 \\ 7 & 1 & 0 & 6 \\ -8 & 0 & 2 & 0 \end{vmatrix} = 0 - (-1)\begin{vmatrix} 3 & 1 & 5 \\ 7 & 1 & 6 \\ -8 & 0 & 0 \end{vmatrix} + 0 - 2\begin{vmatrix} 3 & 1 & 5 \\ 0 & 4 & 0 \\ 7 & 1 & 6 \end{vmatrix}$$

$$= (1)(-8)\begin{vmatrix} 1 & 5 \\ 1 & 6 \end{vmatrix} - 2(4)\begin{vmatrix} 3 & 5 \\ 7 & 6 \end{vmatrix}$$

Example: Cofactor Expansion with Zeros

$$\begin{vmatrix} 3 & 1 & 0 & 5 \\ 0 & 4 & -1 & 0 \\ 7 & 1 & 0 & 6 \\ -8 & 0 & 2 & 0 \end{vmatrix} = 0 - (-1)\begin{vmatrix} 3 & 1 & 5 \\ 7 & 1 & 6 \\ -8 & 0 & 0 \end{vmatrix} + 0 - 2\begin{vmatrix} 3 & 1 & 5 \\ 0 & 4 & 0 \\ 7 & 1 & 6 \end{vmatrix}$$

$$= (1)(-8)\begin{vmatrix} 1 & 5 \\ 1 & 6 \end{vmatrix} - 2(4)\begin{vmatrix} 3 & 5 \\ 7 & 6 \end{vmatrix}$$

$$= -8(1) - 8(-17) = -8(-16)$$

Example: Cofactor Expansion with Zeros

$$\begin{vmatrix} 3 & 1 & 0 & 5 \\ 0 & 4 & -1 & 0 \\ 7 & 1 & 0 & 6 \\ -8 & 0 & 2 & 0 \end{vmatrix} = 0 - (-1)\begin{vmatrix} 3 & 1 & 5 \\ 7 & 1 & 6 \\ -8 & 0 & 0 \end{vmatrix} + 0 - 2\begin{vmatrix} 3 & 1 & 5 \\ 0 & 4 & 0 \\ 7 & 1 & 6 \end{vmatrix}$$

$$= (1)(-8)\begin{vmatrix} 1 & 5 \\ 1 & 6 \end{vmatrix} - 2(4)\begin{vmatrix} 3 & 5 \\ 7 & 6 \end{vmatrix}$$

$$= -8(1) - 8(-17) = -8(-16) = 128\,.$$

Triangular Matrices and Diagonal Matrices

Recall that a square matrix is called *upper triangular* if all of the entries below the main diagonal are zero.

If instead all entries above the main diagonal are zero, it is called *lower triangular*.

It is said to be a diagonal matrix if it is both upper and lower triangular, or equivalently, if the only nonzero entries lie on the main diagonal.

Determinants of Triangular and Diagonal Matrices

Proposition

If A is either upper or lower triangular, or is diagonal, then its determinant is the product of the entries on the main diagonal. That is, if $(\mathrm{A})_{ij} = a_{ij}$ and at least one of the following conditions on the entries holds:

1. $a_{ij} = 0$ *if* $i < j$
2. $a_{ij} = 0$ *if* $i > j$

then $\det(\mathrm{A}) = a_{11}a_{22}\cdots a_{nn} = \prod_{i=1}^{n} a_{ii}$.

Triangular Matrix Example

Proof Sketch.

Compute the determinant via cofactor expansion, each time using a row or column with only one nonzero entry. The resulting computation will be a product of the diagonal entries. □

Example

The determinant of the matrix

$$\begin{bmatrix} -1 & \sqrt{2} & 3-e & \pi \\ 0 & 7 & 667 & 1/97 \\ 0 & 0 & 2 & 5\cosh(1) \\ 0 & 0 & 0 & -4 \end{bmatrix}$$

is $(-1)(7)(2)(-4) = 56$.

Computing a Determinant by Row Reduction

Let $\mathrm{A} \in \mathbb{R}^{n\times n}$ be a matrix, which after one row operation transforms to A'. Then recall:

- If the operation is a row swap $R_i \leftrightarrow R_j$, then $\det(\mathrm{A}') = -\det(\mathrm{A})$.
- If the operation is a row scaling $sR_i \mapsto R_i$, then $\det(\mathrm{A}') = s\det(\mathrm{A})$.
- If the operation is an elementary row replacement $R_i + sR_j \mapsto R_i$ then $\det(\mathrm{A}') = \det(\mathrm{A})$.

So, if we reduce to an upper triangular matrix and keep track of the changes made to the determinant, we can easily calculate $\det(\mathrm{A})$ using far fewer computations than are generally required for a cofactor expansion.

Example: Determinants By Row Reduction

Example

Use Row Reduction to compute the determinant of

$$\mathrm{A} = \begin{bmatrix} 0 & 3 & 5 \\ 2 & 6 & -2 \\ 1 & 1 & -3 \end{bmatrix}$$

The matrix is row equivalent to the matrix

$$\mathrm{B} = \begin{bmatrix} 1 & 1 & -3 \\ 0 & 1 & 1 \\ 0 & 0 & 2 \end{bmatrix}.$$

What row operations do we use to get here? What is $\det \mathrm{B}$?

Example: Determinants By Row Reduction

Example

$$\mathrm{A} = \begin{bmatrix} 0 & 3 & 5 \\ 2 & 6 & -2 \\ 1 & 1 & -3 \end{bmatrix} \sim \begin{bmatrix} 1 & 1 & -3 \\ 0 & 1 & 1 \\ 0 & 0 & 2 \end{bmatrix} = \mathrm{B}$$

We obtain this row equivalence by the operations $R_1 \leftrightarrow R_3$, $R_2 - 2R_1 \mapsto R_1$, $\frac{1}{4}R_2 \mapsto R_2$, and $R_3 - 3R_2 \mapsto R_3$.

Thus $\det \mathrm{B} = \frac{1}{4}(-1) \det \mathrm{A}$. Since $\det \mathrm{B} = (1)(1)(2) = 2$, it follows that

$$\det \mathrm{A} = -8\,.$$

Elementary Matrices

Consider the following three matrices:

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}, \quad \begin{bmatrix} 1 & 0 & 0 \\ 0 & s & 0 \\ 0 & 0 & 1 \end{bmatrix}, \quad \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ s & 0 & 1 \end{bmatrix},$$

where $s \in \mathbb{R}$ is an arbitrary nonzero scalar.

What are the effects of multiplying a 3×3 matrix A by these three matrices on the left? What about on the right?

Realizing Row Operations with Matrix Products

The matrices above each accomplish row operations. We now define three kinds of elementary matrix.

Definition

A matrix $\mathrm{E} \in \mathbb{R}^{n \times n}$ is called en *elementary matrix* if for any given matrix $\mathrm{A} \in \mathbb{R}^{n \times n}$, the effect of left multiplication of A by E is an elementary row operation. Specifically:

i. E is a *row swap matrix* if EA is the matrix obtained by swapping some pair of rows of A,

ii. E is a *row scaling matrix* if EA is the matrix obtained by scaling some row of A,

iii. E is a *simple row combination matrix* if EA is the matrix obtained by adding a multiple of one row of A to another row of A.

Three Types of Elementary Matrices

We can describe the general form of each type of elementary matrix.

Proposition

i. *An* $n \times n$ *row swap matrix* $\mathrm{Swap}(n; i, k)$ *corresponding to the row operation* $R_i \leftrightarrow R_k$ *is obtained from swapping the* i*th and* j*th rows of* I_n*,*

ii. *An* $n \times n$ *row scaling matrix* $\mathrm{Scale}(n; i, s)$ *corresponding to the row operation* $sR_i \mapsto R_i$ *is obtained by replacing the* (i, i)*th entry of* I_n *with* s*,*

iii. *An* $n \times n$ *simple row combination matrix* $\mathrm{Rowadd}(n; i, k, s)$ *corresponding to the row operation* $R_i + sR_k \mapsto R_i$ *is obtained by changing the* (i, k)*th entry of* I_n *to* s*.*

Elementary Matrices and Determinants

It is easy to check, e.g., via a cofactor expansion, that the determinants of elementary matrices are as in the following proposition.

Proposition

i. $\det \mathrm{Swap}(n; i, k) = -1$,
ii. $\det \mathrm{Scale}(n; i, s) = s$,
iii. $\det \mathrm{Rowadd}(n; i, k, s) = 1$.

Multiplicativity of Elementary Matrices

Since we know elementary matrices correspond to row operations, we already know that

- $\det\left(\mathrm{Swap}(n; i, k)\mathrm{A}\right) = -\det(\mathrm{A})$,
- $\det\left(\mathrm{Scale}(n; i, s)\mathrm{A}\right) = s\det(\mathrm{A})$, and
- $\det\left(\mathrm{Rowadd}(n; i, k, s)\mathrm{A}\right) = \det(\mathrm{A})$,

for any matrix $\mathrm{A} \in \mathbb{R}^{n\times n}$.

Reversing the order of the product corresponds to column operations, and so the above equations hold with the order of the products reversed.

Thus, the multiplicativity of determinants holds for elementary matrices.

Inverses from Elementary Matrices

Let $\mathrm{A} \in \mathbb{R}^{n\times n}$ be an $n \times n$ invertible matrix. If A is invertible, then we know $\mathrm{RREF}(\mathrm{A}) = \mathrm{I}_n$, and thus there is some sequence of row operations giving the row equivalence $A \sim \mathrm{I}_n$.

Then one can find a sequence of elementary matrices $\mathrm{E}_1, \mathrm{E}_2, \ldots, \mathrm{E}_k$ such that

$$\mathrm{E}_1\mathrm{E}_2 \cdots \mathrm{E}_k\mathrm{A} = \mathrm{I}_n\,.$$

It follows that $\mathrm{A} = \mathrm{E}_k^{-1} \cdots \mathrm{E}_2^{-1}\mathrm{E}_1^{-1}$ and $\mathrm{A}^{-1} = \mathrm{E}_1\mathrm{E}_2 \cdots \mathrm{E}_k$. Thus, any invertible matrix factors as a product of elementary matrices. Conversely, any product of elementary matrices that doesn't scale a row by 0 is itself an invertible matrix.

For the Second Condition of our Theorem...

Using elementary matrices and the known effects of row operations on determinants as described above, one can prove the multiplicativity of determinants with respect to matrix products, i.e., one can prove that for any matrices $\mathrm{A}, \mathrm{B} \in \mathbb{R}^{n\times n}$

$$\det(\mathrm{AB}) = \det(\mathrm{A})\det(\mathrm{B}) = \det(\mathrm{BA})\,.$$

Multiplicativity from Elementary Matrices

Proof.

Suppose A is invertible.

□

Multiplicativity from Elementary Matrices

Proof.

Suppose A is invertible. Then $\mathrm{A} = \prod_{i=1}^{k} \mathrm{E}_i$ for some elementary matrices $\mathrm{E}_1, \ldots, \mathrm{E}_k$, and

□

Multiplicativity from Elementary Matrices

Proof.

Suppose A is invertible. Then $\mathrm{A} = \prod_{i=1}^{k} \mathrm{E}_i$ for some elementary matrices $\mathrm{E}_1, \ldots, \mathrm{E}_k$, and

$$\det(\mathrm{AB}) = |\mathrm{E}_1 \mathrm{E}_2 \cdots \mathrm{E}_k \mathrm{B}|$$

□

Multiplicativity from Elementary Matrices

Proof.

Suppose A is invertible. Then $\mathrm{A} = \prod_{i=1}^{k} \mathrm{E}_i$ for some elementary matrices $\mathrm{E}_1, \ldots, \mathrm{E}_k$, and

$$\begin{aligned} \det(\mathrm{AB}) &= |\mathrm{E}_1\mathrm{E}_2 \cdots \mathrm{E}_k\mathrm{B}| \\ &= |\mathrm{E}_1||\mathrm{E}_2 \cdots \mathrm{E}_k\mathrm{B}| \end{aligned}$$

□

Multiplicativity from Elementary Matrices

Proof.

Suppose A is invertible. Then $\mathrm{A} = \prod_{i=1}^{k} \mathrm{E}_i$ for some elementary matrices $\mathrm{E}_1, \ldots, \mathrm{E}_k$, and

$$\begin{aligned}\det(\mathrm{AB}) &= |\mathrm{E}_1\mathrm{E}_2\cdots\mathrm{E}_k\mathrm{B}| \\ &= |\mathrm{E}_1||\mathrm{E}_2\cdots\mathrm{E}_k\mathrm{B}| \\ &= |\mathrm{E}_1||\mathrm{E}_2|\cdots|\mathrm{E}_k||\mathrm{E}_B|\end{aligned}$$

□

Multiplicativity from Elementary Matrices

Proof.

Suppose A is invertible. Then $\mathrm{A} = \prod_{i=1}^{k} \mathrm{E}_i$ for some elementary matrices $\mathrm{E}_1, \ldots, \mathrm{E}_k$, and

$$\begin{aligned} \det(\mathrm{AB}) &= |\mathrm{E}_1\mathrm{E}_2 \cdots \mathrm{E}_k\mathrm{B}| \\ &= |\mathrm{E}_1||\mathrm{E}_2 \cdots \mathrm{E}_k\mathrm{B}| \\ &= |\mathrm{E}_1||\mathrm{E}_2| \cdots |\mathrm{E}_k||\mathrm{E}_B| \\ &= |\mathrm{E}_1\mathrm{E}_2 \cdots \mathrm{E}_k| \det(B) \end{aligned}$$

□

Multiplicativity from Elementary Matrices

Proof.

Suppose A is invertible. Then $\mathrm{A} = \prod_{i=1}^{k} \mathrm{E}_i$ for some elementary matrices $\mathrm{E}_1, \ldots, \mathrm{E}_k$, and

$$\begin{aligned} \det(\mathrm{AB}) &= |\mathrm{E}_1\mathrm{E}_2 \cdots \mathrm{E}_k\mathrm{B}| \\ &= |\mathrm{E}_1||\mathrm{E}_2 \cdots \mathrm{E}_k\mathrm{B}| \\ &= |\mathrm{E}_1||\mathrm{E}_2| \cdots |\mathrm{E}_k||\mathrm{E}_B| \\ &= |\mathrm{E}_1\mathrm{E}_2 \cdots \mathrm{E}_k| \det(B) \\ &= \det(A)\det(B)\,. \end{aligned}$$

□

Multiplicativity from Elementary Matrices

Proof.

Suppose A is invertible. Then $\mathrm{A} = \prod_{i=1}^{k} \mathrm{E}_i$ for some elementary matrices $\mathrm{E}_1, \ldots, \mathrm{E}_k$, and

$$\begin{aligned} \det(\mathrm{AB}) &= |\mathrm{E}_1\mathrm{E}_2 \cdots \mathrm{E}_k\mathrm{B}| \\ &= |\mathrm{E}_1||\mathrm{E}_2 \cdots \mathrm{E}_k\mathrm{B}| \\ &= |\mathrm{E}_1||\mathrm{E}_2| \cdots |\mathrm{E}_k||\mathrm{E}_B| \\ &= |\mathrm{E}_1\mathrm{E}_2 \cdots \mathrm{E}_k| \det(B) \\ &= \det(A)\det(B)\,. \end{aligned}$$

On the other hand, if A is not invertible, then AB is also not invertible, whence both $\det(\mathrm{AB})$ and $\det(A)\det(B)$ are zero. □

Orientations

We will now define orientation for an ordered collection of n linearly independent vectors in $\mathbb{R}^n$ via determinants.

Definition

An ordered collection $(\mathbf{v}_1, \dots \mathbf{v}_n)$ of linearly independent vectors in $\mathbb{R}^n$ is said to be *positively oriented* if and only if $\det \begin{bmatrix} \mathbf{v}_1 & \dots & \mathbf{v}_n \end{bmatrix} > 0$, otherwise it is said to be negatively oriented.

A negatively oriented collection $(\mathbf{v}_1, \dots \mathbf{v}_n)$ becomes positively oriented after performing a transposition of elements (changing the order by swapping two vectors). More generally, if R is a reflection, then $(\mathrm{R}\mathbf{v}_1, \dots, \mathrm{R}\mathbf{v}_n)$ has the opposite orientation from $(\mathbf{v}_1, \dots \mathbf{v}_n)$.

Signed Volume

Recall

- For vectors $\mathbf{a}_1, \mathbf{a}_2 \in \mathbb{R}^2$, $\det\left(\left[\begin{array}{cc} \mathbf{a}_1 & \mathbf{a}_2 \end{array}\right]\right)$ is *signed area* of the parallelogram spanned by $\mathbf{a}_1$ and $\mathbf{a}_2$, with positive sign if and only if the $\mathbf{a}_1$ rotates through the parallelogram counterclockwise towards $\mathbf{a}_2$,
- For vectors $\mathbf{a}_1, \mathbf{a}_2, \mathbf{a}_3 \in \mathbb{R}^3$ $\det\left(\left[\begin{array}{ccc} \mathbf{a}_1 & \mathbf{a}_2 & \mathbf{a}_3 \end{array}\right]\right)$ is the *signed volume* of the parallelepiped spanned by $\mathbf{a}_1$, $\mathbf{a}_2$, and $\mathbf{a}_3$, with positive sign if and only if the triple $(\mathbf{a}_1, \mathbf{a}_2, \mathbf{a}_3)$ is right-handed.

We want to interpret the determinant of an $n \times n$ matrix similarly as *signed hypervolume* for an appropriate object, with sign determined by the orientation of the set of vectors (i.e., by their order).

Signed Hypervolume

Definition

Given a set $\mathbf{a}_1, \ldots, \mathbf{a}_n \in \mathbb{R}^n$ of n linearly independent vectors, the *n-dimensional parallelotope spanned by* $\mathbf{a}_1, \ldots, \mathbf{a}_n$ is the set $\mathscr{P} = \mathscr{P}(\mathbf{a}_1, \ldots, \mathbf{a}_n) := \{\sum_{i=1}^n x_i \mathbf{a}_i \,|\, 0 \le x_i \le 1,\ i = 1, \ldots, n\} \subset \mathbb{R}^n$.

The *n-dimensional volume*, also called *hypervolume*, or simply *volume* of this parallelotope is $\mathcal{V}(\mathscr{P}) := \left|\det\left(\begin{bmatrix} \mathbf{a}_1 & \ldots & \mathbf{a}_n \end{bmatrix}\right)\right|$.

The determinant $\det\left(\begin{bmatrix} \mathbf{a}_1 & \ldots & \mathbf{a}_n \end{bmatrix}\right)$ is geometrically the signed hypervolume of the parallelotope spanned by $\mathbf{a}_1, \ldots, \mathbf{a}_n$. It is positive if and only if $(\mathbf{a}_1, \ldots, \mathbf{a}_n)$ is positively oriented.

Area Example: 2D

Example

Express the area of a regular hexagon inscribed in the unit circle in $\mathbb{R}^2$, using determinants.

Solution: We can express a regular hexagon inscribed in the unit circle as the *convex hull* of 6 points along the circle, equally separated by angles of $2\pi/6 = \pi/3$.

By placing one vertex at $(1,0)$, we can build the hexagon with 6 triangles, each congruent to the triangle with vertices $\mathbf{0}$, $\mathbf{e}_1$ and $(1/2)\mathbf{e}_1 + (\sqrt{3}/2)\mathbf{e}_2 = \cos(\pi/3)\mathbf{e}_1 + \sin(\pi/3)\mathbf{e}_2$.

The area of this triangle is $1/2$ the area of the parallelogram spanned by $\mathbf{e}_1$ and $(1/2)\mathbf{e}_1 + (\sqrt{3}/2)\mathbf{e}_2$, so we compute

$$\mathcal{A} = 6\left(\frac{1}{2}\right)\begin{vmatrix} 1 & 1/2 \\ 0 & \sqrt{3}/2 \end{vmatrix} = \frac{3\sqrt{3}}{2}.$$

Volume Example: Parallelpiped

Example

Find the volume of the parallelepiped spanned by the vectors $-2\mathbf{e}_1 + 2\mathbf{e}_2 - \mathbf{e}_3$, $\mathbf{e}_1 + \mathbf{e}_2 + 2\mathbf{e}_3$ and $2\mathbf{e}_1 - 2\mathbf{e}_2 - \mathbf{e}_3$.

Solution: we compute the determinant

$$\begin{vmatrix} -2 & 1 & 2 \\ 2 & 1 & -2 \\ -1 & 2 & -1 \end{vmatrix} = 2 + 2 + 8 - (-2) - (8) - (-2) = 8\,.$$

Here, the 3×3 determinant was calculated using the Scheme of Sarrus:

$$\begin{vmatrix} \vec{a}_1 & \mathbf{a}_2 \mathbf{a}_3 \end{vmatrix} = a_{11}a_{22}a_{33} + a_{12}a_{23}a_{31} + a_{13}a_{21}a_{32}$$
$$- a_{31}a_{22}a_{13} - a_{32}a_{23}a_{11} - a_{33}a_{21}a_{12}\,.$$

Tetrahedra

Challenge Problem: A tetrahedron is a solid with four vertices, six edges, and four triangular faces. You should convince yourself that if a tetrahedron has edges $\mathbf{a}_1, \mathbf{a}_2, \mathbf{a}_3 \in \mathbb{R}^3$, then its volume is given by

$$\frac{1}{6}\left|\det\left(\begin{bmatrix} \mathbf{a}_1 & \mathbf{a}_2 & \mathbf{a}_3 \end{bmatrix}\right)\right| .$$

Tesseract (A.K.A. Hypercube)

Definition

A *tesseract*, also known as a *hypercube*, an *8-cell*, a *regular octachoron*, or several other names beloved by sci-fi officianados, is region of 4-dimensional space, analogous to a cube, spanned by 4 orthogonal vectors of equal length.

Tesseracts can be visualized by projecting them into 3-dimensional space, and understood as the result of taking 8 cubes and gluing them together in a way that generalizes how a cube is constructed from a net of squares.

Definition

The *standard unit hypercube* is the tesseract defined by $\{\mathbf{x} \in \mathbb{R}^4 \,|\, 0 \leq x_i \leq 1 \text{ for } i = 1, 2, 3, 4\}$. It is said to be *spanned* by the standard unit basis vectors $\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3, \mathbf{e}_4$.

Definition

A *parallelochoron* or a 4-*parallelotope with parallelepiped faces* is the broader class of volumetric forms within $\mathbb{R}^4$ defined as a set by
$\{x_1\mathbf{a}_1 + x_2\mathbf{a}_2 + x_3\mathbf{a}_3 + x_4\mathbf{a}_4 \,|\, 0 \leq x_i \leq 1 \text{ for } i = 1, 2, 3, 4\}$
for some some linearly independent quartet of 4-vectors $\mathbf{a}_1$, $\mathbf{a}_2$, $\mathbf{a}_3$, and $\mathbf{a}_4$. Thus, a parallelochoron is the result of applying an invertible transformation $T(\mathbf{x}) = \begin{bmatrix} \mathbf{a}_1 & \mathbf{a}_2 & \mathbf{a}_3 & \mathbf{a}_4 \end{bmatrix} \mathbf{x}$ to the standard unit hypercube, and can be said to be the parallelochoron *spanned by* $\mathbf{a}_1$, $\mathbf{a}_2$, $\mathbf{a}_3$, and $\mathbf{a}_4$.

Example: Computing hypervolume

Example

Observe that the vectors

$$\mathbf{v}_1 = \begin{bmatrix} 1/2 \\ 1/2 \\ 1/2 \\ 1/2 \end{bmatrix}, \mathbf{v}_2 = \begin{bmatrix} -1/2 \\ 1/2 \\ 1/2 \\ -1/2 \end{bmatrix}, \mathbf{v}_3 = \begin{bmatrix} 1/2 \\ -1/2 \\ 1/2 \\ -1/2 \end{bmatrix}, \mathbf{v}_4 = \begin{bmatrix} 1/2 \\ 1/2 \\ -1/2 \\ -1/2 \end{bmatrix}$$

have the same lengths and dot pairwise to 0, and thus span a tesseract in $\mathbb{R}^4$.

a. Find the hypervolume of this tesseract.

b. Is the quartet $(\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3, \mathbf{v}_4)$ positively oriented?

Example: Computing hypervolume

Example

c. Find hypervolume of the above tesseract obtained by transforming it via the matrix

$$A = \begin{bmatrix} 0 & -1 & 2 & 0 \\ 3 & 0 & \sqrt{2} & \sqrt{3} \\ 0 & -\sqrt{2} & 0 & -\sqrt{3} \\ -3 & 0 & 2 & 0 \end{bmatrix}.$$

Computing Hypervolume

Example

Solutions:

a. Via the row operations $R_3 - R_2 \mapsto R_3$, $R_4 - R_1 \mapsto R_4$, $R - 1 + R_2 \mapsto R_1$, $R_2 - \frac{1}{2}R_1 \mapsto R_2$, and $R_4 + R_3 \mapsto R_4$, one obtains the row equivalence

$$\begin{bmatrix} \mathbf{v}_1 & \mathbf{v}_2 & \mathbf{v}_3 & \mathbf{v}_4 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 0 & 1 \\ 0 & 1/2 & -1/2 & -1/2 \\ 0 & 0 & 1 & -1 \\ 0 & 0 & 0 & -2 \end{bmatrix}.$$

Since this matrix is upper triangular, its determinant is the product of diagonal entries, namely, $(1)(1/2)(1)(-2) = -1$. Since the only row operations used are row additions preserving the determinant, we conclude the volume of the tesseract is 1.

Computing Hypervolume

Example

Solutions:

b. From part a. we know the determinant $\begin{vmatrix} \mathbf{v}_1 & \mathbf{v}_2 & \mathbf{v}_3 & \mathbf{v}_4 \end{vmatrix} = -1 < 0$, so the quartet $(\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3, \mathbf{v}_4)$ is negatively oriented.

c. The new volume $\mathcal{V}'$ will equal $|\det(\mathrm{A})|$ times the volume $\mathcal{V} = 1$ of the original tesseract, so it suffices to compute the absolute value of $\det(\mathrm{A})$. Given the presence of 7 zeros among the 16 entries, this is a good candidate for cofactor expansion, e.g. along the first row.

Computing Hypervolume

Example

c. **Solutions**:

$$\mathcal{V}' = \left\| \begin{matrix} 0 & -1 & 2 & 0 \\ 3 & 0 & \sqrt{2} & \sqrt{3} \\ 0 & -\sqrt{2} & 0 & -\sqrt{3} \\ -3 & 0 & 2 & 0 \end{matrix} \right\|$$

$$= \left| -(-1)\begin{vmatrix} 3 & \sqrt{2} & \sqrt{3} \\ 0 & 0 & -\sqrt{3} \\ -3 & 2 & 0 \end{vmatrix} + 2\begin{vmatrix} 3 & 0 & \sqrt{3} \\ 0 & -\sqrt{2} & -\sqrt{3} \\ -3 & 0 & 2 \end{vmatrix} \right|$$

$$= \left| \sqrt{3}\begin{vmatrix} 3 & \sqrt{2} \\ -3 & 2 \end{vmatrix} + 2(-\sqrt{2})\begin{vmatrix} 3 & \sqrt{3} \\ -3 & 0 \end{vmatrix} \right|$$

$$= \left|\sqrt{3}(6+3\sqrt{2}) - 2\sqrt{2}(3\sqrt{3})\right| = 6\sqrt{3} - 3\sqrt{6}\,.$$

Determinant as a Stretch Factor

Proposition

If $T : \mathbb{R}^n \to \mathbb{R}^n$ is a bijective linear transformation represented by a matrix A, then

- *the volume $\mathcal{V}\big(T(\mathscr{P})\big)$ of the image parallelotope $T(\mathscr{P})$ of a parallelotope $\mathscr{P} \subset \mathbb{R}^n$ is equal to the product of the volume $\mathcal{V}(\mathscr{P})$ times $|\det(\mathrm{A})|$,*
- *The transformation is orientation preserving if and only if $\det(\mathrm{A}) > 0$, so $\det(\mathrm{A})\mathcal{V}(\mathscr{P}) = \mathcal{V}\big(T(\mathscr{P})\big)$ if and only if $\det(\mathrm{A}) > 0$.*

Cramer's Rule

Proposition

Let $\mathrm{A} \in \mathbb{R}^{n\times n}$ *be an invertible square matrix. Suppose* $\mathbf{x}$ *is the unique solution to a system* $\mathrm{A}\mathbf{x} = \mathbf{b}$*. If* $\mathbf{x} = x_1\mathbf{e}_1 + \ldots + x_i\mathbf{e}_i + \ldots + x_n\mathbf{e}_n$*, then*

$$x_i = \frac{\det(\mathrm{A}_{\mathbf{b},i})}{\det(\mathrm{A})}$$

where $\mathrm{A}_{\mathbf{b},i}$ *is the matrix obtained by replacing the* i*th column with* $\mathbf{b}$*.*

Proof Sketch.

Simply compute $\det(\mathrm{A}_{\mathbf{b},i})$ using multilinearity of the determinant in the columns, and show that it equals $x_i \det(\mathrm{A})$. □

The Adjugate Formula for Inverses

Definition

The *adjugate matrix* A^* of a square matrix A is the matrix whose (i,j)th entry is $(-1)^{i+j}\det(\mathrm{A}_{ij})$, i.e., the entries are the signed cofactors of A.

Corollary

If $\mathrm{A} \in \mathbb{R}^{n\times n}$ *is an invertible matrix, then*

$$\mathrm{A}^{-1} = \frac{1}{\det(\mathrm{A})}(A^*)^{\tau}.$$

Challenge Problem: Write up the details of the proof of Cramer's rule, and then prove the corollary giving the inverse formula in terms of the adjugate.

Use the adjugate formula to confirm the inverse formula for 2×2 matrices, and then write out an explicit inverse formula for a generic 3×3 matrix in terms of its entries (a_{ij}).

Homework

- MyMathLab for section 2.3 is due on Tuesday, March 6.
- MyMathLab for section 3.1 is due on Thursday, March 8.
- Please read sections 3.1, 3.2, and 3.3 of the text (Cramer's Rule is optional, Volumes are not).
- Spring Break begins the weekend of March 10 (i.e., the weekend after next!)