CHALLENGE PROBLEMS

A. HAVENS

This is a list of challenge and extra credit problems, which may count towards quiz or homework deficits. You may try any problems of your choosing, and you need not do all parts of a problem to receive some credit (though you should still try all parts of any problem you select).

1. Early Material

The problems in this section were proposed early in the course, and thus I require that you solve them (first, at least) with only the tools available at the time, namely, linear systems, the Gauss-Jordan row reduction algorithm, and any relevant theorems from before the introduction of cross products, determinants, and general vector space theory. Note that you may assume elementary results about orthogonality and the dot product.

(1) For the following problem, you must *not* use cross products; you should however check that the expression you obtain for the resulting normal is equivalent to the cross product up to scaling.

   (a) Given arbitrary 3-vectors $\mathbf{u}$ and $\mathbf{v}$ spanning a plane through a point $\mathbf{x}_0 \in \mathbb{R}^3$, find the equation of the plane, and express a line through $\mathbf{0}$ normal to the plane as the solution of some homogeneous system, in terms of the components of $\mathbf{u}$ and $\mathbf{v}$.

   (b) Describe the normal line to this plane through a given point $\mathbf{y} \in \mathbb{R}^3$ parametrically.

(2) Without using determinants, show that three planes in $\mathbb{R}^3$ intersect in a unique point if and only if their normals span $\mathbb{R}^3$.

(3) Rigorously prove the following statements:

   (a) For any finite set S of vectors in $\mathbb{R}^n$, if $\mathbf{v} \in S$ is a vector such that
   $$\operatorname{span} S = \operatorname{span}\left(S - \{\mathbf{v}\}\right),$$
   then S is a linearly *dependent* set.

   (b) If a finite set S' also containing $\mathbf{v}$ is linearly *independent*, then
   $$\operatorname{span}\left(S' - \{\mathbf{v}\}\right) \subsetneq \operatorname{span} S'.$$

(4) For the following, you should attempt to work solely with the definitions and results introduced in lecture 8 on linear independence and dependence.

(a) Describe an algorithm to reduce a linearly dependent set $S \subset \mathbb{R}^n$ of finitely many vectors to a linearly independent set $S' \subset \mathbb{R}^n$ such that $\operatorname{span} S = \operatorname{span} S'$. Show that regardless of any choices made in the algorithm, the final number of vectors in S' will be the same, and depends only on $\operatorname{span} S$ itself (and not on S or choices you made).

(b) Explain why the number $|S'|$ of vectors in the linearly independent set S' must be less than or equal to n.

(5) Show that the map $\mathscr{R}_\theta(\mathbf{x}) = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix} \mathbf{x}$ is a counter-clockwise rotation of $\mathbb{R}^2$ about $\mathbf{0}$ by an angle of $\theta \in [0, 2\pi)$.

(6) For each of the following geometric transformations, give a general matrix representing the map.

(a) Projection onto a line $\ell = \operatorname{span}\{\mathbf{u}\} \subset \mathbb{R}^3$

(b) Reflection through a line $\ell = \operatorname{span}\{\mathbf{u}\} \subset \mathbb{R}^3$

(c) Orthogonal projection onto the plane $ax_1 + bx_2 + cx_3 = 0$

(d) Reflection across the plane $ax_1 + bx_2 + cx_3 = 0$

(e) Rotation of $\mathbb{R}^3$ by an angle φ about an axis $\ell = \operatorname{span}\{\mathbf{u}\}$ (see problem (14) in section 4 below on spatial rotations and the Rodriguez formula).

(7) Fix an angle $\theta \in [0, 2\pi)$. Let R_θ represent planar rotation about $\mathbf{0}$ by the angle θ, and let

$$\mathrm{M}_1 = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}.$$

(a) By computing the product $\mathrm{R}_\theta \mathrm{M}_1 \mathrm{R}_\theta^{-1}$, show that the reflection through a line $\ell = \operatorname{span}\left\{ \begin{bmatrix} \cos\theta \\ \sin\theta \end{bmatrix} \right\}$ is given by

$$\mathrm{Ref}_\ell(\mathbf{x}) = \begin{bmatrix} \cos 2\theta & \sin 2\theta \\ \sin 2\theta & -\cos 2\theta \end{bmatrix} \mathbf{x}\,.$$

Compare with the formula obtained by computing $\mathrm{Ref}_\ell(\mathbf{x}) = (2\mathrm{proj}_\ell - \mathrm{I}_2)\mathbf{x}$.

(b) Let $z = x_1 + ix_2$ be the complex number associated to the vector $\mathbf{x} = x_1\mathbf{e}_1 + x_2\mathbf{e}_2$. By using Euler's identity $e^{i\theta} = \cos\theta + i\sin\theta$, show that one can rewrite the reflection through ℓ using the conjugation map $z \mapsto \bar{z} = x_1 - ix_2$. In particular, show that the reflection through $\ell = \{se^{i\theta} : s \in \mathbb{R}\}$ is given by

$$z \mapsto e^{i\theta}\overline{(e^{-i\theta}z)} = e^{2i\theta}\bar{z}\,.$$

(8) This problem deals with facts about left and write inverses of general functions. Let X and Y be sets, and consider an abstract function $f : X \to Y$.

(a) Prove that a function $f : X \to Y$, , has a left inverse g if and only if it is injective (one-to-one). Show that any left inverse g of f, if it exists, is surjective.

(b) Analogously, show that f is right invertible if and only if f is surjective (onto), and that any right inverse h of f, if it exists, is injective.

(c) Show that f is (totally) invertible if and only if it is bijective, i.e., both injective and surjective.

(9) Give necessary and sufficient conditions for an $m \times n$ matrix $\mathrm{A} \in \mathbb{R}^{m\times n}$ to be left or right invertible. Describe an algorithm to compute either a left or right inverse, presuming one exists, or to determine otherwise that none exists.

2. Mid-Course Material

This section contains problems on material covered between the two midterms, excluding the rank-nullity theorem.

(1) This set of problems expands upon the details of the cross product–determinant connection. For these problems, use the following geometric definition of the cross product:

Definition. For $\mathbf{u}, \mathbf{v} \in \mathbb{R}^3$, the cross product is the vector defined by the following conditions:

i. $\mathbf{u} \times \mathbf{v} = \mathbf{0}$ whenever $\mathbf{u}$ and $\mathbf{v}$ are parallel.

ii. For $\mathbf{u}$ and $\mathbf{v}$ linearly independent,

$$\mathbf{u} \times \mathbf{v} := \mathcal{A}(\mathscr{P})\hat{\mathbf{n}}_{\mathscr{P}} = \|\mathbf{u}\|\|\mathbf{v}\| \sin(\theta)\, \hat{\mathbf{n}}_{\mathscr{P}}\,,$$

where $\mathscr{P} := \{t_1\mathbf{u} + t_2\mathbf{v} \,|\, t_1, t_2 \in [0,1]\}$ is the parallelogram in $\mathbb{R}^3$ with vertices $\mathbf{0}$, $\mathbf{u}$, $\mathbf{v}$ and $\mathbf{u}+\mathbf{v}$, $\hat{\mathbf{n}}_{\mathscr{P}}$ is a unit normal to $\mathscr{P}$ such that $(\mathbf{u}, \mathbf{v}.\hat{\mathbf{n}}_{\mathscr{P}})$ is right handed, $\mathcal{A}(\mathscr{P})$ is the area of $\mathscr{P}$, and $\theta \in (0, \pi)$ is the (positive) angle through $\mathscr{P}$ between the vectors $\mathbf{u}$ and $\mathbf{v}$.

(a) Use the geometric definition to prove the following proposition giving the coordinate method of computing the cross product.

Proposition. *The three dimensional cross product is the* 3*-vector valued product given by*

$$(\mathbf{u}, \mathbf{v}) \mapsto \mathbf{u} \times \mathbf{v} = \begin{vmatrix} u_2 & u_3 \\ v_2 & v_3 \end{vmatrix} \mathbf{e}_1 - \begin{vmatrix} u_1 & u_3 \\ v_1 & v_3 \end{vmatrix} \mathbf{e}_2 + \begin{vmatrix} u_1 & u_2 \\ v_1 & v_2 \end{vmatrix} \mathbf{e}_3\,,$$

where $\mathbf{u} = u_1\mathbf{e}_1 + u_2\mathbf{e}_2 + u_3\mathbf{e}_3$ *and* $\mathbf{v} = v_1\mathbf{e}_1 + v_2\mathbf{e}_2 + v_3\mathbf{e}_3$.

(b) Rigorously show that the triple product $\mathbf{u} \cdot (\mathbf{v} \times \mathbf{w})$ is equal to the determinant $\begin{vmatrix} \mathbf{u} & \mathbf{v} & \mathbf{w} \end{vmatrix}$.

(c) Rigorously verify the following claim.

Claim. The vector triple product of $\mathbf{u}, \mathbf{v}, \mathbf{w} \in \mathbb{R}^3$ computes the volume of the *parallelepiped*

$$\{x_1\mathbf{u} + x_2\mathbf{v} + x_3\mathbf{w} \,|\, 0 \le x_1, x_2, x_3 \le 1\} \subset \mathbb{R}^3\,.$$

(d) Find a matrix representing the linear map $\mathbf{u}\times : \mathbb{R}^3 \to \mathbb{R}^3$ that maps $\mathbf{v}$ to $\mathbf{u} \times \mathbf{v}$.

(e) Verify that the following properties of the cross product hold without appealing to coordinate calculations or brute force.

- $(s\mathbf{u}) \times \mathbf{v} = s(\mathbf{u} \times \mathbf{v}) = \mathbf{u} \times (s\mathbf{v})$,
- $\mathbf{u} \times \mathbf{v} = -(\mathbf{v} \times \mathbf{u})$,
- $\mathbf{u} \times \mathbf{u} = \mathbf{0}$,
- $\mathbf{u} \times (\mathbf{v} + \mathbf{w}) = \mathbf{u} \times \mathbf{v} + \mathbf{u} \times \mathbf{w}$,
- $\mathbf{u} \cdot (\mathbf{v} \times \mathbf{w}) = \mathbf{w} \cdot (\mathbf{u} \times \mathbf{v})$,
- $\mathbf{u} \cdot (\mathbf{v} \times \mathbf{w}) = -\mathbf{v} \cdot (\mathbf{u} \times \mathbf{w})$.

(2) A tetrahedron is a solid with four vertices, six edges, and four triangular faces. Rigorously demonstrate that if a tetrahedron has adjacent edges $\mathbf{a}_1, \mathbf{a}_2, \mathbf{a}_3 \in \mathbb{R}^3$ sharing the vertex $\mathbf{0}$, then its volume is given by

$$\frac{1}{6}\left|\det\left(\begin{bmatrix} \mathbf{a}_1 & \mathbf{a}_2 & \mathbf{a}_3 \end{bmatrix}\right)\right| .$$

(3) Recall that Cramer's Rule is the following proposition regarding the use of determinants in solving systems.

Proposition. *Let* $\mathrm{A} \in \mathbb{R}^{n\times n}$ *be an invertible square matrix. Suppose* $\mathbf{x}$ *is the unique solution to a system* $\mathrm{A}\mathbf{x} = \mathbf{b}$*. If* $\mathbf{x} = x_1\mathbf{e}_1 + \ldots + x_i\mathbf{e}_i + \ldots + x_n\mathbf{e}_n$*, then*

$$x_i = \frac{\det(\mathrm{A}_{\mathbf{b},i})}{\det(\mathrm{A})}$$

where $\mathrm{A}_{\mathbf{b},i}$ *is the matrix obtained by replacing the* i*th*-column *of* A *with* $\mathbf{b}$.

(a) Write up a careful proof of Cramer's rule.

(b) Recall that the definition of the classical adjugate matrix.

Definition. The **adjugate matrix** A^* of a square matrix A is the matrix whose (i, j)th entry is $(-1)^{i+j}\det(\mathrm{A}_{ij})$, i.e., the entries are the signed cofactors of A.

Prove the following corollary of Cramer's Rule.

Corollary. *If* $\mathrm{A} \in \mathbb{R}^{n\times n}$ *is an invertible matrix, then*

$$\mathrm{A}^{-1} = \frac{1}{\det(\mathrm{A})}(A^*)^{\mathrm{t}} .$$

(c) Use the adjugate formula to confirm the inverse formula for 2×2 matrices, and then write out an explicit inverse formula for a generic 3×3 matrix in terms of its entries (a_{ij}).

(4) Let V and W be $\mathbb{F}$-vector spaces, and $T : V \to W$ any linear map. Prove that the image $T(V) \subseteq W$ is an $\mathbb{F}$-vector subspace of W.

(5) Let U and W be subspaces of an $\mathbb{F}$-vector space V.

(a) Show that $U \cap W = \{\mathbf{y} \in V \,|\, \mathbf{y} \in U \text{ and } \mathbf{y} \in W\}$ is a subspace.

(b) Show that $U \cup W = \{\mathbf{y} \in V \,|\, \mathbf{y} \in U \text{ or } \mathbf{y} \in W\}$ need not be a subspace.

(c) Show that $U + W = \{\mathbf{y} \in V \,|\, \mathbf{y} = \mathbf{u} + \mathbf{w}, \mathbf{u} \in U, \mathbf{w} \in W\}$ is a subspace, and it is the *minimal subspace* containing $U \cup W$, in the sense that any other subspace of V containing $U \cup W$ must also contain $U + W$.

(d) Argue that for general subspace U, W of an $\mathbb{F}$-vector space V, the following holds:

$$\dim_{\mathbb{F}} V = \dim_{\mathbb{F}} U + \dim_{\mathbb{F}} W - \dim_{\mathbb{F}}(U \cap W) .$$

(e) Suppose $U \cap W = \{\mathbf{0}\}$. Show that $\dim_{\mathbb{F}} U + \dim_{\mathbb{F}} W = \dim_{\mathbb{F}} V$

3. Late-Course Material

This section contains problems on material after the second midterm, but before inner product spaces and orthogonality.

(1) Recall the general rank nullity theorem.

Theorem (General Rank-Nullity Theorem)**.** *Let V be a finite dimensional $\mathbb{F}$-vector space, and let $T : V \to W$ be a linear map. Then*

$$\boxed{\dim_{\mathbb{F}} V = \dim_{\mathbb{F}} T(V) + \dim_{\mathbb{F}} \ker T\,.}$$

Use the theory of linear coordinates to give an alternate proof of this theorem (recall, in class we gave a proof by constructing a basis tailored to the transformation, which extends a basis of the kernel to one of the domain, and shows that the images of the non-kernel vectors in the extended basis are a basis for the image).

(2) Recall that for a finite directed graph $\mathcal{G} = (\mathcal{V}, \mathcal{E})$ with $|\mathcal{V}| = m$ and $|\mathcal{E}| = n$ we can define an **incidence matrix** as follows:

Definition. The **incidence matrix of** $\mathcal{G}$ is the $m \times n$ matrix $\mathrm{I}(\mathcal{G})$ whose (i, j)-th entry is equal to -1 if the j-th edge leaves the i-th vertex, $+1$ if the j-th edge enters the i-th vertex, and 0 if the j-th edge is not incident with the i-th vertex. If $\mathcal{G}$ is undirected, then the (i, j)-th entry is 1 if and only if the j-th edge is incident with the ith vertex, and zero otherwise.

Recall also the terminology below.

Definition. A **walk** on a graph is an alternating sequence of vertices and edges initiated and terminating in a vertex, with any consecutive vertex-edge or edge-vertex pair incident. Thus, a walk can be specified by a sequence of **coincident** edges.

Definition. A walk is called a **trail** if there are no repeated edges.

Definition. A **cycle** is a walk which returns to the initial vertex, and a **simple cycle** is a cycle which is also a trail.

Definition. If $\mathcal{G}$ is directed, then a simple cycle of $\mathcal{G}$ is associated to a vector $\mathbf{x} \in \mathbb{R}^n$, called a **weight vector**, with components $x_i \in \{+1, -1\}$, such that $\mathrm{I}(\mathcal{G})\mathbf{x} = \mathbf{0}$; an edge has positive weight if the cycle traverses it according to its orientation, and is negative if the edge is traversed against its orientation. Simple cycles on a (not necessarily directed) graph are said to be **independent** if the corresponding weight vectors are linearly independent (for some orientations of edges).

Finally, recall the proposition:

Proposition. *Let* $\mathrm{I}(\mathcal{G})$ *be the incidence matrix of a finite directed graph* $\mathcal{G} = (\mathcal{V}, \mathcal{E})$ *with* m *vertices and* n *edges.*

(*i.*) *The rank* $\operatorname{rank}\mathrm{I}(\mathcal{G}) = m - k$ *where* k *is the number of connected components of* $\mathcal{G}$*. In particular, for a connected graph,* $k = 1$*, and* $\operatorname{rank}\mathrm{I}(\mathcal{G}) = m - 1$.

(*ii.*) *If* $\mathcal{G}$ *is connected, then its left null space is spanned by the vector* $\mathbf{e}_1 + \mathbf{e}_2 + \ldots + \mathbf{e}_m$*. In particular, the column space of* $\mathrm{I}(\mathcal{G})$ *is the set of all* $\mathbf{x} \in \mathbb{R}^m$ *perpendicular to* $\mathbf{e}_1 + \mathbf{e}_2 + \ldots + \mathbf{e}_m$*, and so the sum of components of such* $\mathbf{x}$ *must be zero.*

(*iii.*) *The null space* $\operatorname{Nul}\mathrm{I}(\mathcal{G})$ *consists of vectors* $\mathbf{x} \in \mathbb{R}^n$*, whose components, thought of as currents on the corresponding edges, yield a solution to Kirchhoff's current law in the absence of a current source: in matrix form the law reads* $\mathrm{A}\mathbf{x} = \mathbf{0}$*. In particular, the dimension of the null space is the maximum number of independent simple cycles in* $\mathcal{G}$*.*

Prove the proposition.

(3) Recall Euler's formula for connected graphs:

$$\max(\#\text{ of indep. simple cycles}) - (\#\text{ of edges}) + (\#\text{ of vertices}) = 1\,.$$

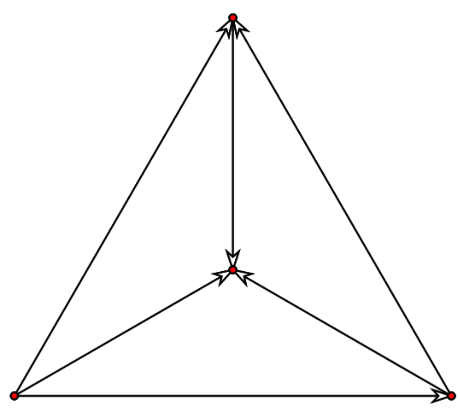

Figure 1. The directed graph
$\mathcal{G} = \big(\,\{v_0, v_1, v_2, v_3\}\,, \{(v_0, v_1), (v_0, v_2), (v_0, v_3), (v_1, v_2), (v_1, v_3), (v_2, v_3)\}\,\big)\,.$

(a) For the graph in the figure above, write down an incidence matrix, and find bases for the column and null spaces.

(b) Verify by hand that there are at most 3 independent cycles by using the picture.

(c) What if the orientations are altered? Will there always be 3 independent cycles?

(d) What would an acyclic graph look like?

(e) Construct some graphs with 0, 1, 2, and 11 cycles, and check Euler's graph formula for each.

(4) Recall the definition of an outer product of vectors.

Definition. Given two vectors $\mathbf{u} \in \mathbb{R}^m$ and $\mathbf{v} \in \mathbb{R}^n$, their outer product is the matrix $m \times n$

$$\mathbf{u} \otimes \mathbf{v} := \mathbf{u}\mathbf{v}^{\mathrm{t}} = \begin{bmatrix} u_1v_1 & u_1v_2 & \dots & u_1v_n \\ u_2v_1 & u_2v_2 & \dots & u_2v_n \\ \vdots & \vdots & \ddots & \vdots \\ u_mv_1 & u_mv_2 & \dots & u_mv_n \end{bmatrix}.$$

Show that every matrix which can be written as an outer product of two vectors has rank at most 1. When is the rank 0?

(5) Recall the following definition.

Definition. An n-th order recurrence relation is a discrete relation of the form

$$x_k = f(x_{k-n}, x_{k-n+1}, \dots x_{k-1}),$$

for integers $k \geq n$ where f is some function. An n-th order recurrence is **linear homogeneous** if f is a homogeneous linear function, i.e., if the recurrence relation is of the form

$$x_k = a_0x_{k-n} + a_1x_{k-n+1} + \dots + a_{n-1}x_{k-1} = \sum_{i=0}^{n-1} a_i x_{k-n+i},$$

for numbers $a_0, \dots, a_{n-1}$.

Consider a general homogeneous n-th order linear recurrence of the form

$$x_n = a_0x_0 + \dots a_{n-1}x_{n-1}.$$

Recall that there is associated to such a system a **companion matrix**: writing

$$\mathbf{x}_k := \begin{bmatrix} x_{k-n+1} \\ x_{k-n+2} \\ \vdots \\ x_{k-1} \\ x_k \end{bmatrix}, \quad \mathrm{C} := \left[\begin{array}{c|c} \mathbf{0} & \mathrm{I}_n \\ \hline \multicolumn{2}{c}{\mathbf{a}^{\mathrm{t}}} \end{array}\right] = \begin{bmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \ddots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 0 & 1 \\ a_0 & a_1 & \cdots & \cdots & a_{n-1} \end{bmatrix},$$

one can equivalently specify the recurrence by the first order matrix vector recurrence $\mathbf{x}_k = \mathrm{C}\mathbf{x}_{k-1}$.

(a) Show that the polynomial $t^n - \sum_{i=0}^{n-1} a_i t^i$ is the characteristic polynomial of the associated companion matrix C.

(b) For any eigenvalue λ which is a root of order m of the characteristic polynomial, show that $x_k = k^p\lambda^k$, is a solution of the recurrence equation for any $p \in \{0, 1, \dots m-1\}$

(c) Show that the general solution is given by linear combinations of terms of the form $k^p\lambda^k$. That is, show that any solution x_k of the recurrence has form

$$\begin{aligned} x_k &= \sum_{i=1}^{l}\sum_{j=1}^{m_l} b_{i,j}k^{j-1}\lambda_i^k \\ &= \lambda_1^k(b_{1,0} + b_{1,1}k + \ldots b_{1,m_1}k^{m_1-1}) \\ &\qquad\qquad + \ldots \lambda_l^k(b_{l,0} + b_{l,1}k + \ldots b_{l,m_l}k^{m_l-1}) \ , \end{aligned}$$

where $\lambda_1 \ldots \lambda_l$ are distinct eigenvalues of C with respective algebraic multiplicities $m_1, \ldots, m_l$, and $b_{i,j}$ are constants.

(d) If one specifies values for $x_0, \ldots x_{n-1}$, does this uniquely determine the constants $b_{i,j}$?

4. Bitter-End-of-Course Material

This section contains problems on the dot product, inner products, and orthogonality.

(1) Recall the proposition giving properties of the Euclidean dot product

Proposition. *Let $\mathbf{u}, \mathbf{v}, \mathbf{w} \in \mathbb{R}^n$ be any real n-vectors, and $s, t \in \mathbb{R}$ be any scalars. The Euclidean dot product $(\mathbf{u}, \mathbf{v}) \mapsto \mathbf{u} \cdot \mathbf{v}$ satisfies the following properties.*

*(i.) The dot product is **symmetric**: $\mathbf{u} \cdot \mathbf{v} = \mathbf{v} \cdot \mathbf{u}$.*

*(ii.) The dot product is **bilinear**:*

- $(s\mathbf{u}) \cdot \mathbf{v} = s(\mathbf{u} \cdot \mathbf{v}) = \mathbf{u} \cdot (s\mathbf{v})$,
- $(\mathbf{u} + \mathbf{v}) \cdot \mathbf{w} = \mathbf{u} \cdot \mathbf{w} + \mathbf{v} \cdot \mathbf{w}$.

Thus in particular, for fixed $\mathbf{w}$, the maps $\mathbf{x} \mapsto \mathbf{w} \cdot \mathbf{x}$ and $\mathbf{x} \mapsto \mathbf{x} \cdot \mathbf{w}$ are linear maps valued in $\mathbb{R}$.

*(iii.) The dot product is **positive definite**: $\mathbf{u} \cdot \mathbf{u} \geq 0$ with equality if and only if $\mathbf{u} = \mathbf{0}$.*

(a) Give a proof of this proposition using the coordinate formula

$$(\mathbf{u}, \mathbf{v}) = \left(\sum_{i=1}^{n} u_i \mathbf{e}_i, \sum_{i=1}^{n} v_i \mathbf{e}_i \right) \mapsto \sum_{i}^{n} u_i v_i \,.$$

(b) Give an alternate proof of each property above, assuming instead only the following **geometric definition** of a dot product.

Definition. For $\mathbf{u}, \mathbf{v} \in \mathbb{R}^n$, with $\theta \in [0, \pi]$ the measure of the angle of separation between the vectors $\mathbf{u}$ and $\mathbf{v}$, as measured in a plane containing both $\mathbf{u}$ and $\mathbf{v}$, the dot product of $\mathbf{u}$ and $\mathbf{v}$ is the scalar

$$\mathbf{u} \cdot \mathbf{v} = \|\mathbf{u}\| \|\mathbf{v}\| \cos\theta \,,$$

where $\|\cdot\|$ is the usual Euclidean norm.

(2) A regular tetrahedron is a solid in $\mathbb{R}^3$ with four faces, each of which is an equilateral triangle. Find the angles between the faces of a tetrahedron, which are **dihedral angles** (a dihedral angle is an angle between the faces of a polyhedron).

(3) By a diagonal of a cube, we mean the line segment from one vertex of a cube to the farthest vertex across the cube. By a diagonal of a cube's face, we mean the diagonal of the square face from one vertex to the opposite vertex of that face.

(a) Find the lengths of the diagonals of a cube and diagonals of faces in terms of the side length of a cube.

(b) Find the angles between a diagonal of a cube and an adjacent edge of the cube.

(c) Each diagonal of the cube is adjacent to how many face diagonals? Find the angle between a diagonal of a cube and an adjacent face diagonal.

(4) Prove that, for any $\mathbf{u}, \mathbf{v} \in \mathbb{R}^n$,

$$2\left\|\mathbf{u}\right\|^2 + 2\left\|\mathbf{v}\right\|^2 = \left\|\mathbf{u}+\mathbf{v}\right\|^2 + \left\|\mathbf{u}-\mathbf{v}\right\|^2 \text{ , and}$$

$$\mathbf{u}\cdot\mathbf{v} = \frac{1}{4}\Big(\left\|\mathbf{u}+\mathbf{v}\right\|^2 - \left\|\mathbf{u}-\mathbf{v}\right\|^2 \Big)$$

(5) Recall the definition of a norm on a vector space V.

Definition. A norm on an $\mathbb{F}$-vector space V is a function $\|\cdot\| : V \to \mathbb{F}$ such that the following properties hold for all $\mathbf{u}, \mathbf{v} \in V$ and $s \in \mathbb{F}$:
(i.) **non-degeneracy**: $\|\mathbf{u}\| \geq \mathbf{0}$ with equality if and only if $\mathbf{u} = \mathbf{0}$,
(ii.) **absolute homogeneity**: $\|s\mathbf{u}\| = |s|\|\mathbf{u}\|$, where $|\cdot|$ is a norm on the field $\mathbb{F}$,
(iii.) **sub-additivity**: $\|\mathbf{u}+\mathbf{v}\| \leq \|\mathbf{u}\| + \|\mathbf{v}\|$.

Prove that any normed vector space $(V, \|\cdot\|)$ can be made into an inner product space with an inner product $\langle\cdot,\cdot\rangle : V \times V \to \mathbb{F}$ such that $\|\mathbf{u}\|^2 = \langle\mathbf{u},\mathbf{u}\rangle$.

(6) Consider linearly independent vectors $\mathbf{u}, \mathbf{v} \in \mathbb{R}^2$, and let $\mathscr{P}$ be the parallelogram whose sides they span. Under what conditions are the diagonals of $\mathscr{P}$ orthogonal?

(7) Demonstrate via vector algebra that the diagonals of a parallelogram always bisect each other.

(8) Prove the following proposition.

Proposition. *Let $W \subset \mathbb{R}^n$ be any subspace. Then $W^\perp$ is a subspace of $\mathbb{R}^n$. Moreover, the following statements are equivalent:*
(a) $\mathbf{y} \in W^\perp$,
(b) $\mathbf{y}\cdot\mathbf{v} = 0$ *for every* $\mathbf{v} \in S$ *where* $W = \operatorname{span} S$,
(c) $\mathbf{y} \in \operatorname{Nul} \mathrm{B}^{\mathrm{t}}$ *for any matrix* B *whose columns are an orthogonal basis of* W,
(d) $\|\mathbf{y}-\mathbf{w}\| = \|\mathbf{y}+\mathbf{w}\|$ *for every* $\mathbf{w} \in W$,
(e) $\|\mathbf{y}-\mathbf{w}\|$ *is minimized if and only if* $\mathbf{w} = \mathbf{0}$

(9) Prove the following theorem, which is in essence an addendum to the Rank-Nullity theorem (taken together, they form what Gilbert Strang calls **the fundamental theorem of linear algebra**).

Theorem. *Let* $\mathrm{A} \in \mathbb{R}^{m\times n}$. *Then the orthogonal complement* $(\operatorname{Row} \mathrm{A})^\perp$ *of the row space* $\operatorname{Row}\mathrm{A}$ *is naturally isomorphic to the null space* $\operatorname{Nul}\mathrm{A}$ *(via the isomorphism* $\operatorname{Row}\mathrm{A} \cong \operatorname{Col}\mathrm{A}^{\mathrm{t}}$*), and the left nullspace* $\operatorname{Nul}\mathrm{A}^{\mathrm{t}}$ *is equal to to the orthogonal complement* $(\operatorname{Col}\mathrm{A})^\perp$ *of the column space* $\operatorname{Col}\mathrm{A}$.

(10) Give a formula for a reflection through a subspace $W \subseteq \mathbb{R}^n$ using projection.

(11) Let $W \subset \mathbb{R}^n$ be a proper nontrivial subspace. Consider the sequence of maps

$$\mathbf{0} \to W \to \mathbb{R}^n \to W^\perp \to \mathbf{0}\,,$$

where the first two arrows are given by inclusion, the map to $W^\perp$ is given by projection, and the final map is the trivial map.

(a) Show that the image of each map is the kernel of the next map. Such a chain of maps, where the kernel of each map is the image from the previous map, is called an **exact sequence**, and in this case, where there are four maps and the first and last spaces of the sequence are both trivial, the sequence is called a **short exact sequence**.

(b) For a general short exact sequence of linear maps of vector spaces

$$\mathbf{0} \to V \to W \to U \to \mathbf{0}\,,$$

argue that the first two arrows are injective maps, and the last two arrows are surjective maps. Show that $W \cong V \oplus U$, i.e., that every $\mathbf{w} \in W$ can be written uniquely as a sum of elements such that one is in the image of the map coming from V and the other is in the complement of the kernel of the map from W.

(12) Prove the following proposition.

Proposition. *Let* $\{\mathbf{u}_1, \dots, \mathbf{u}_p\}$ *be an orthonormal basis of a subspace* $W \subseteq \mathbb{R}^n$ *and let* $\mathrm{U} = \begin{bmatrix} \mathbf{u}_1 & \dots \mathbf{u}_p \end{bmatrix}$. *Then for any* $\mathbf{y} \in \mathbb{R}^n$

$$\mathrm{proj}_{\,W}(\mathbf{y}) = \mathrm{U}\mathrm{U}^{\mathrm{t}}\mathbf{y} = (\mathrm{U} \otimes \mathrm{U})\mathbf{y}\,.$$

(13) In analogy to the subspace test for vector spaces and using the previous problem as your starting model, devise and prove a **subgroup test** to determine whether a subset H of a group G is itself a group with respect to the operation it inherits from G.

(14) In this exercise you will derive a general matrix formula for spatial rotations, called the **Rodriquez formula**, and then explore the structure of SO(3). Fix a unit axial vector $\hat{\mathbf{u}} \in \mathbb{R}^n$ and an angle $\varphi \in [0, 2\pi)$.

(a) Show that the rotation of a vector $\mathbf{x}$ about the axis span $\{\hat{\mathbf{u}}\}$ by an angle of φ counterclockwise (relative to the view from the tip of $\hat{\mathbf{u}}$ towards $\mathbf{0}$) is given by

$$\mathscr{R}_\varphi^{\hat{\mathbf{u}}}(\mathbf{x}) = \big(1 - \cos(\varphi)\big)\mathrm{proj}_{\,\hat{\mathbf{u}}}(\mathbf{x}) + \cos(\varphi)\mathbf{x} + \sin(\varphi)\hat{\mathbf{u}} \times \mathbf{x}\,.$$

This is the Rodriguez formula for spatial rotation.

(b) Use the preceding part to write out a matrix U such that $\mathscr{R}_\varphi^{\hat{\mathbf{u}}}(\mathbf{x}) = \mathrm{U}\mathbf{x}$, in terms of the components of $\hat{\mathbf{u}}$ and the angle φ.

(c) What are the eigenvalues and complex eigenvectors of the matrix U associated to $\mathscr{R}_\varphi^{\hat{\mathbf{u}}}(\mathbf{x})$ (hint: do not try to compute the characteristic polynomial directly from the matrix).

(15) Show that the elements of O(3) that are not in SO(3) decompose as products of a reflection matrix for reflection though a plane and a matrix in SO(3). Show that any matrix in O(3) decomposes as a product of just reflection matrices for reflections through a system of planes. What's the maximum number of reflections needed?

(16) Let $\big(V, \langle\cdot,\cdot\rangle\big)$ be an inner product space. Let $\|\cdot\|$ be the 2-norm induced by the inner product $\langle\cdot,\cdot\rangle$. Use the Cauchy-Schwarz inequality to prove the triangle inequality

$$\|\mathbf{u}-\mathbf{v}\| \le \|\mathbf{u}\| + \|\mathbf{v}\|\,.$$

(17) Give another argument for the Cauchy-Schwarz inequality, using the idea of best approximation in a subspace.

(18) Verify that the set

$$\left(\frac{1}{\sqrt{2}}, \sin t, \cos t, \sin 2t, \cos 2t, \dots, \sin nt, \cos nt\right)$$

is an orthonormal basis of the space of real trigonometric polynomials $\mathscr{T}_n$ of degree $\le n$ with the inner product structure given by

$$\langle f, g\rangle = \frac{1}{\pi}\int_{-\pi}^{\pi} f(t)g(t)\,\mathrm{d}t\,.$$

In particular, you must show that this set spans $\mathscr{T}_n$ as well as showing the orthonormality of the elements.