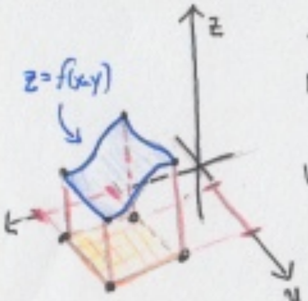
A. Havens: Notes on Double integrals

1.

Assume temporarily that $f(x,y)$ is continuous and positive over some rectangle

$$R=[a,b]\times[c,d]=\{(x,y)\in\mathbb{R}^2 \mid a\le x\le b,\ c\le y\le d\}.$$

$z=f(x,y) \rightsquigarrow$ graph surface above R, the "roof" of a "room" above R, with walls regions of planes $x=a$, $x=b$, $y=c$, & $y=d$.



We want to define $\iint_R f(x,y)\,dA$ to be the Volume V of this room. Observe that we can approximate the volume by a collection of boxes, stretching from R to $z=f(x,y)$.

2.

Setup for Riemann Sums:

Approximate V by a sum of box Volumes $V_{ij}=\mathcal{A}(R_{ij})\cdot f(x_i^*,y_j^*)$ where $\mathcal{A}(R_{ij})$ = area of a subrectangle, and $(x_i^*,y_j^*)\in R_{ij}$ is a sample point determining the box height.

Building the approximation: fix integers $m,n>0$. Set $\Delta x=\frac{b-a}{m}$, $\Delta y=\frac{d-c}{n}$

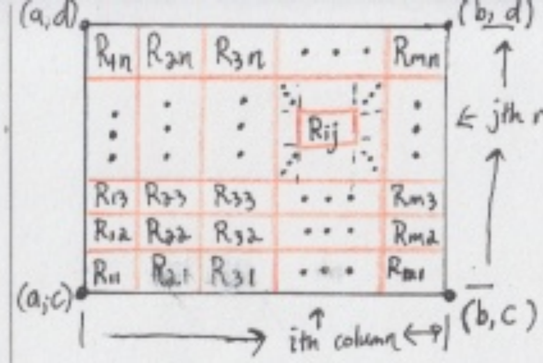
$x_0:=a,\quad x_i:=x_{i-1}+\Delta x=a+i\Delta x$

$y_0:=c,\quad y_j:=y_{j-1}+\Delta y=c+j\Delta y$

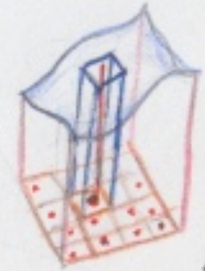
$R_{ij}:=[x_{i-1},x_i]\times[y_{j-1},y_j]$

= Subrectangle in ith column and jth row from $R_{11}=[a,a+\Delta x]\times[c,c+\Delta y]$.

3.



Choose $x_i^*\in[x_{i-1},x_i]$ and $y_j^*\in[y_{j-1},y_j]$.



The value $f(x_i^*,y_j^*)$ determines the height of the box over R_{ij}.

$$V_{ij}=f(x_i^*,y_j^*)\cdot\Delta x\Delta y.$$

4.

Thus $V\approx \sum_{i=1}^{m}\sum_{j=1}^{n} f(x_i^*,y_j^*)\underbrace{\Delta x\Delta y}_{\Delta A}$

Def^{n}: Let $f(x,y)$ be a function defined over the Rectangle $R=[a,b]\times[c,d]$. Then the double integral of f over R is

$$\iint_R f(x,y)\,dA=\lim_{m,n\to\infty}\sum_{i=1}^{m}\sum_{j=1}^{n} f(x_i^*,y_j^*)\Delta A,$$

provided this limit exists.

Note: for continuous f over R, $\iint_R f(x,y)\,dA$ always exists.

5.

Remarks:

1.) Note we did not assume $f(x,y)\ge 0$ for the definition. If $f(x,y)\ge 0$ over R, then $\iint_R f(x,y)\,dA$ may be interpreted as a measure of volume between $R\subset\mathbb{R}^2 \underset{\text{as plane } z=0}{\xhookrightarrow{\text{embed}}} \mathbb{R}^3$ and the graph $G_f(R)\subseteq\mathbb{R}^3$.

If f takes on negative values, then we count the volume negatively when G_f is below $z=0$. Letting V_+ be the volume above $z=0$ and below G_f, and V_- the volume below $z=0$ and above G_f,

6

we may interpret the double integral as net volume:

$$\iint_R f(x,y)\,dA=V_+-V_-.$$

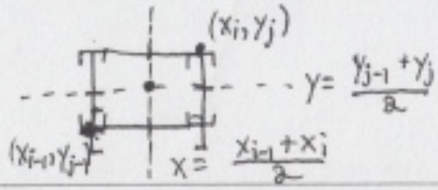
A surface such as $z=x^2-y^2$ may have net volume 0 above a rectangle centered on $(0,0)\in\mathbb{R}^2\hookrightarrow\mathbb{R}^3$

7.

2.) We can decently approximate $\iint_R f(x,y)\,dA$ without too many rectangles if we use the <u>midpoint</u> <u>rule</u>:

$$\iint_R f(x,y)\,dA \approx \sum_{i=1}^{m}\sum_{j=1}^{n} f(\bar{x}_i,\bar{y}_j)\,\Delta x\Delta y$$

where $(\bar{x}_i,\bar{y}_j) = \left(\tfrac{1}{2}(x_{i-1}+x_i), \tfrac{1}{2}(y_{j-1}+y_j)\right)$ is the midpoint of R_{ij}.



8.

Exercise: Generalize the trapezoid rule to a "tangent approximation method" for computing $\iint_R f(x,y)\,dA$ using solids whose tops (bottoms) are the portions of tangent planes to $(\bar{x}_i, \bar{y}_j, f(\bar{x}_i,\bar{y}_j))$ over (under) rectangles R_{ij}.

3.) If $f(x,y) = 1$ over R, then

$$\iint_R f(x,y)\,dA = \iint_R 1\,dA = A(R)$$
$$= (b-a)\cdot(d-c).$$

9.

Q: Given explicit f & R, how do we compute $\iint_R f(x,y)\,dA$ in practice?

Answer: <u>Iterated</u> partial integrals.

Partial integration is akin to partial differentiation — integrate with respect to one variable, while holding the other(s) constant.

There are definite and indefinite versions.

10.

<u>Def^n</u>: The indefinite partial integral of $f(x,y)$ with respect to x is the class of general antiderivatives, represented as

$$\int f(x,y)\,dx = F(x,y) + C(y)$$

where $\frac{\partial C}{\partial x} = 0$ & $\frac{\partial F}{\partial x} = f(x,y)$.

By the fundamental theorem of calculus, we may take $F(x,y)$ to be an antiderivative obtained through definite integration

11.

$$F(x,y) = \int_a^x f(t,y)\,dt.$$

Indeed, $\frac{\partial F}{\partial x}(x,y) = \frac{\partial}{\partial x}\int_a^x f(t,y)\,dt$

$$= f(x,y)$$

by FTC I.

We also have FTC II for partial integrals. Eg.

$$\int_a^b f(x,y)\,dx = F(b,y) - F(a,y)$$
$$\int_c^d g(x,y)\,dy = G(x,d) - G(x,c).$$

12

Example: $\int_0^2 x+y\,dy = \left[xy + \frac{y^2}{2}\right]_0^2$
$= 2x + 2 - 0$
$= 2x + 2.$

Thus note a definite partial integral may in general be a function of the remaining (unintegrated) variables.

Example: $\int_0^\pi y\sin(xy)\,dx$
$= \left[-\cos(xy)\right]_0^\pi = 1 + \cos(\pi y).$

13.

Let S be the solid bounded between the planes $z=0$, $x=a$, $x=b$, $y=c$, $y=d$, and the graph $z=f(x,y)$.
Assume f is continuous over $R=[a,b]\times[c,d]$.

Slice S by the plane $x=x_0$; observe that the area of this slice is $A(x_0)=\int_c^d f(x_0,y)\,dy$.

Thus, definite partial integrals have a geometric interpretation as the area of a slice of a solid, with the other variable(s) held constant.

14.

By the cross-sectional area principle:

$$V := \mathrm{Vol}(S) = \int_a^b A(x)\,dx$$

$$(*)\qquad = \int_a^b \left[\int_c^d f(x,y)\,dy\right] dx,$$

provided $\iint_R f(x,y)\,dA$ exists and the partial integral exists.

$(*)$ thus expresses $V=\iint_R f(x,y)\,dA$ as an <u>iterated integral</u>.

15.

Note: There is no reason a priori to prefer to slice first along planes $x=x_0$. If instead we use cross-sectional areas in slices along constant y, we obtain

$$V = \iint_R f(x,y)\,dA = \int_c^d \left[\int_a^b f(x,y)\,dx\right] dy$$

We just need to be sure that f is "nice enough" over R to ensure that both iterated integrals exist and agree.

16.

<u>Fubini's Theorem</u>: If f is integrable over R and both iterated integrals exist, then

$$\iint_R f(x,y)\,dA = \int_a^b\int_c^d f(x,y)\,dy\,dx$$
$$= \int_c^d\int_a^b f(x,y)\,dx\,dy.$$

In particular, if f is continuous, or, bounded with "nice enough" discontinuities (e.g., jump discontinuities on finitely many curves), and both iterated integrals exist.

17.

(Hard) exercise: Let $R=[0,1]\times[0,1]$ and define

$$f(x,y) = \begin{cases} 1 & \text{if } x \text{ is rational} \\ 2y & \text{if } x \text{ is irrational} \end{cases}$$

$$g(x,y) = \begin{cases} \dfrac{x^2-y^2}{(x^2+y^2)^2} & \text{if } (x,y)\neq(0,0) \\ 0 & \text{if } (x,y)=(0,0). \end{cases}$$

Explain in each case why Fubini's Theorem does not apply. Determine precisely how the theorem's conclusions fail in each case, by computing iterated integrals where possible.

18.

<u>Example</u>: We compute $\iint_R y\sin(xy)\,dA$ for the rectangle $R=[1,2]\times[0,\pi]$, using iterated integration.

We have two possible orders to set up the iterated integrals:

$$\int_1^2\int_0^\pi y\sin(xy)\,dy\,dx \quad \text{or}$$

$$\int_0^\pi\int_1^2 y\sin(xy)\,dx\,dy.$$

One order is easier (no integration by parts!)

19.

Integrating with respect to x involves only a simple substitution $u = xy \rightsquigarrow du = y\,dx$, so we have

$$\iint_R y\sin(xy)\,dA = \int_0^{\pi}\int_1^2 y\sin(xy)\,dx\,dy$$

$$= \int_0^{\pi}\Big[-\cos(xy)\Big]_1^2\,dy$$

$$= \int_0^{\pi} \cos y - \cos 2y \; dy$$

$$= \Big[\sin y - \tfrac{1}{2}\sin 2y\Big]_0^{\pi}$$

$$= 0.$$

20.

Average value of a function over a rectangle:

Definition: The average value of $f(x,y)$ on the rectangle $R = [a,b]\times[c,d]$ is

$$\bar{f}(R) := \frac{1}{A(R)}\iint_R f(x,y)\,dA$$

$$= \frac{1}{(b-a)(d-c)}\iint_{[a,b]\times[c,d]} f(x,y)\,dA.$$

21

Example: We compute the average value of $f(x,y) = xy^2$ over $R = [0,2]\times[0,3]$.

$$\bar{f}(R) = \frac{1}{2\cdot 3}\iint_R xy^2\,dA$$

$$= \frac{1}{6}\int_0^2\int_0^3 xy^2\,dy\,dx$$

$$= \frac{1}{6}\int_0^2 \frac{xy^3}{3}\Big|_0^3\,dx$$

$$= \frac{1}{6}\int_0^2 9x\,dx = \frac{1}{6}\Big[\frac{9x^2}{2}\Big]_0^2 = 3.$$

22.

A trick:

$$\int_a^b\int_c^d f(x)g(y)\,dy\,dx$$

$$= \left(\int_a^b f(x)\,dx\right)\cdot\left(\int_c^d g(y)\,dy\right).$$

e.g. for $\bar{f}(R)$ as above:

$$6\bar{f}(R) = \left(\int_0^2 x\,dx\right)\left(\int_0^3 y^2\,dy\right)$$

$$= 2\cdot 9.$$

23.

Some bounds:

If $m \le f(x,y) \le M$ for all $(x,y)\in R$, and f integrable over R, then

$$m\,A(R) \le \iint_R f(x,y)\,dA \le M\,A(R)$$

or $m \le \bar{f}(R) \le M$.

In particular, one can take

$m = \min\{f(x,y) \mid (x,y)\in R\}$ and

$M = \max\{f(x,y) \mid (x,y)\in R\}$, f cts on R.

24.

e.g. Estimate $\iint_R \underbrace{\ln(2xy - x - y + 2)}_{f(x,y)}\,dA$ on $R = [0,1]\times[0,1]$.

One can apply the techniques of the previous chapter to show that

$m = \min_R f(x,y) = 0$ (occuring @ $(1,0)$ and $(0,1)$)

$M = \max_R f(x,y) = \ln 2$ (occuring @ $(0,0)$ and $(1,1)$)

Thus $0 \le \iint_R \ln(2xy - x - y + 2)\,dA \le \ln 2$ since $A(R) = 1$.

Integrals over general Regions. 1.

Assume $D \subseteq \mathbb{R}^2$ compact (closed and bounded) and $f(x,y)$ continuous on D. Let R be a bounding rectangle for D.

Let $F(x,y) = \begin{cases} f(x,y) & \text{if } (x,y) \in D \\ 0 & \text{if } (x,y) \notin D, \end{cases}$

and define $\iint_D f(x,y)\,dA = \iint_R F(x,y)\,dA$.

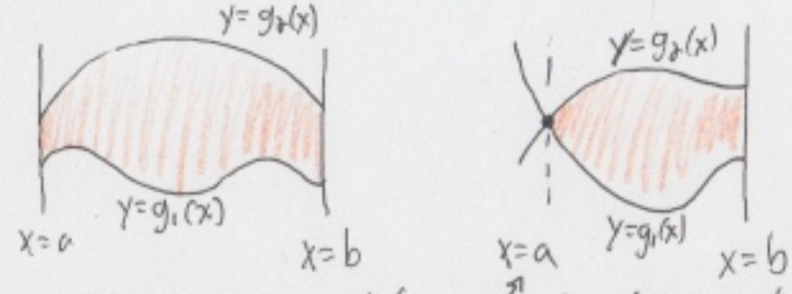
To compute such integrals using iterated integrals, we define two types of simple regions.

2.

Definition: A region D is said to be of type one if there exists a pair of functions $g_1, g_2 : [a,b] \to \mathbb{R}$ such that

$$D = \{(x,y) \in \mathbb{R}^2 \mid a \le x \le b,\ g_1(x) \le y \le g_2(x)\}$$

Type I regions illustrated:



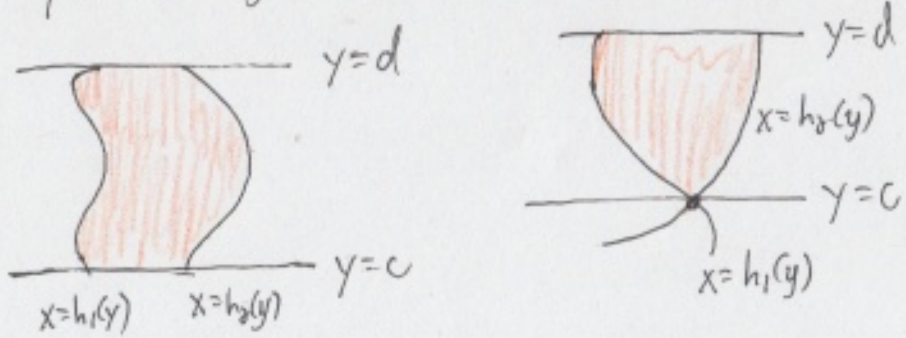
observe that a left or right boundary may be an intersection point for the upper and lower curves.

3.

Similarly, a region E is Type II if

$$E = \{(x,y) \in \mathbb{R}^2 \mid h_1(y) \le x \le h_2(y),\ c \le y \le d\}$$

Type II regions illustrated



4.

Proposition: If D is a type I region as above, and E a type II region, and $f(x,y)$ is continuous on each, then

$$\iint_D f(x,y)\,dA = \int_a^b \int_{g_1(x)}^{g_2(x)} f(x,y)\,dy\,dx$$

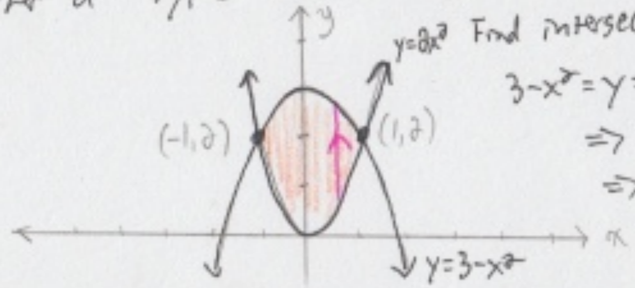
$$\iint_E f(x,y)\,dA = \int_c^d \int_{h_1(y)}^{h_2(y)} f(x,y)\,dx\,dy.$$

Think: Type I bounds: $\int_{\text{left line}}^{\text{right line}} \int_{\text{bottom curve}}^{\text{top curve}}$

Type II bounds: $\int_{\text{bottom line}}^{\text{top line}} \int_{\text{left curve}}^{\text{right curve}}$

Example: Find the volume of the solid beneath $z=6x^2y$ above the region bounded by $y=3-x^2$ and $y=2x^2$. 5

- Step 1: Draw the region and determine bounds as either a type I or type II region.



Find intersection points:

$$3-x^2=y=2x^2$$
$$\Rightarrow 3x^2=3$$
$$\Rightarrow x=\pm 1$$

Region is a type I: $D=\{(x,y)\in\mathbb{R}^2 \mid -1\le x\le 1,\ 2x^2\le y\le 3-x^2\}$

Step II: Setup iterated integral with bounds. 6

$$\iint_D z(x,y)\,dA=\int_{-1}^{1}\int_{2x^2}^{3-x^2} 6x^2y\,dy\,dx$$

Step III: Compute!

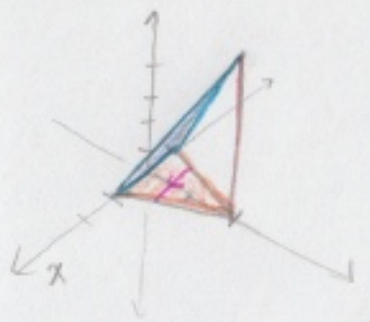
$$=\int_{-1}^{1} 3x^2y^2\Big|_{2x^2}^{3-x^2}\,dx$$
$$=\int_{-1}^{1} 3x^2\left[(3-x^2)^2-(2x^2)^2\right]dx$$
$$=\int_{-1}^{1} 3x^2\left[9-6x^2+x^4-4x^4\right]dx$$
$$=\int_{-1}^{1} 27x^2-18x^4-9x^6\,dx$$

Note that every term of the integrand is even, whence 7

$$\iint_D z(x,y)\,dA=2\int_0^1 27x^2-18x^4-9x^6\,dx$$
$$=2\left[9x^3-\frac{18}{5}x^5-\frac{9}{7}x^7\right]_0^1$$
$$=2\left(9-\frac{18}{5}-\frac{9}{7}\right)=2\left(\frac{144}{35}\right)=\frac{288}{35}.$$

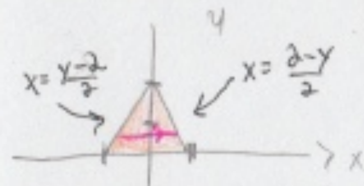
Let's do another example

Find the Volume of the Solid bounded by $z=0$, $z=2y$, $2x+y=2$ and $-2x+y=2$. 8



The solid is a tetrahedron, with base a triangle in the xy plane bounded by $y=0$, $y=2+2x$, and $y=2-2x$.

To avoid splitting the integral into two pieces, we can treat the triangle as a type two region



$$T=\left\{(x,y) \,\middle|\, \frac{y-2}{2}\le x\le \frac{2-y}{2},\ 0\le y\le 2\right\}.$$

Thus, the tetrahedron's volume is 9.

$$\iint_T 2y\,dA = \int_0^2 \int_{\frac{y-2}{2}}^{\frac{2-y}{2}} y\,dx\,dy$$

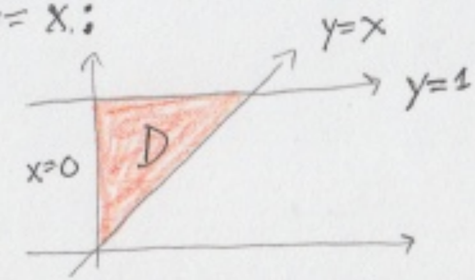
$$= \int_0^2 \frac{1}{2}\left(2y - y^2 - y^2 + 2y\right) dy$$

$$= \frac{1}{2}\int_0^2 4y - 2y^2\,dy$$

$$= \frac{1}{2}\left[2y^2 - \frac{2}{3}y^3\right]_0^2 = \frac{16}{3} - 4 = \frac{4}{3}.$$

When a region can be expressed as either type I or type II, it can be advantageous to choose an order of integration and bounds so as to avoid difficult antiderivatives. 10.

Example: Let D be the region bounded by the y axis, the line $y=1$, and the line $y=x$:



Compute

$$\int_0^1 \int_x^1 \sin(y^2)\,dy\,dx = \iint_D \sin(y^2)\,dA.$$

The integral is initially expressed for D as a type I region — but, we don't know a closed form antiderivative of $\sin(y^2)$! 11.

Rewriting for D as a type II:

$$D = \{(x,y) \mid 0 \le y \le 1,\ 0 \le x \le y\}$$

$$\iint_D \sin(y^2)\,dA = \int_0^1 \int_0^y \sin(y^2)\,dx\,dy$$

$$= \int_0^1 x\sin(y^2)\Big|_0^y\,dy$$

$$= \int_0^1 y\sin(y^2)\,dy.$$

A simple substitution of $y^2 = u$, $2y = du$ now would allow us to complete the evaluation. We could also write 12.

$$\iint_D \sin(y^2)\,dA = \int_0^1 \frac{1}{2}\,d\left(-\cos(y^2)\right)$$

$$= \frac{-\cos(y^2)}{2}\Big|_0^1$$

$$= \frac{1}{2}\left(1 - \cos(1)\right).$$

Properties of Double integrals.

Assume f, g both integrable over any regions involved.

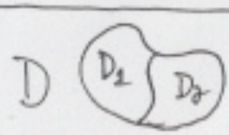
1.) $\iint_D c f(x,y)\, dA = c \iint_D f(x,y)\, dA$.

2.) $\iint_D [f(x,y) + g(x,y)]\, dA = \iint_D f(x,y)\, dA + \iint_D g(x,y)\, dA$

3.) Suppose $f(x,y) \geq g(x,y)$ throughout D. Then

$$\iint_D f(x,y)\, dA \geq \iint_D g(x,y)\, dA.$$

4.) $\iint_D 1\, dA = A(D)$



5.) If $D = D_1 \sqcup D_2$: $\iint_D f(x,y)\, dA = \iint_{D_1} f(x,y)\, dA + \iint_{D_2} f(x,y)\, dA$

Area between curves and Volumes between graph surfaces.

- Appealing to (4.) above, under the assumption that $g_2(x) \geq g_1(x)$ for $x \in [a,b]$, if $D = \{(x,y) \mid a \leq x \leq b,\ g_1(x) \leq y \leq g_2(y)\}$,

$$\iint_D dA = A(D) = \int_a^b \int_{g_1(x)}^{g_2(x)} dy\, dx$$

$$= \int_a^b g_2(x) - g_1(x)\, dx,$$

which recovers the simplest case of the Calculus II formula for area between graphs.

A similar idea works for the volume between surfaces. Suppose $z_+(x,y)$ and $z_-(x,y)$ are functions over D, defining surfaces such that $z_+(x,y) \geq z_-(x,y)$ over D. Then the volume between is

$$V = \iint_D z_+ - z_-\, dA.$$

Lets use this to set up the double integral for volume of a sphere, treating $D = \{(x,y) \mid x^2 + y^2 \leq R^2\}$ as a type I region.

$z_+ = \sqrt{R^2 - x^2 - y^2}$

$z_- = -\sqrt{R^2 - x^2 - y^2}$

$D = \{(x,y) \mid -R \leq x \leq R,\ -\sqrt{R^2-x^2} \leq y \leq \sqrt{R^2-x^2}\}$

$$V = \int_{-R}^{R} \int_{-\sqrt{R^2-x^2}}^{\sqrt{R^2-x^2}} z_+ - z_-\, dy\, dx$$

$$= \int_{-R}^{R} \int_{-\sqrt{R^2-x^2}}^{\sqrt{R^2-x^2}} 2\sqrt{R^2 - x^2 - y^2}\, dy\, dx.$$

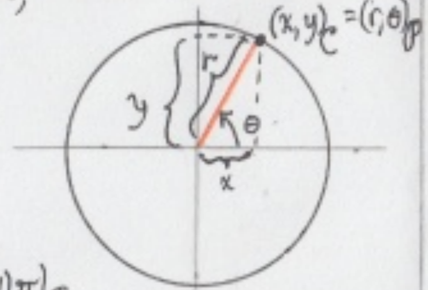
Computing this in the present form requires trigonometric substitution. Alternatively, we can change to <u>polar coordinates</u>.

Double integrals in Polar Coordinates

1.

Review of Polar Coordinates:

Let $(x,y)_C$ denote the Cartesian/rectangular coordinate of a point in $\mathbb{R}^2$. We can associate new coordinates $(r,\theta)_P$, called polar coordinates, such that



$$\left\{\begin{matrix} r^2 = x^2+y^2 \\ x\tan\theta = y \end{matrix}\right\} \longleftrightarrow \left\{\begin{matrix} x = r\cos\theta \\ y = r\sin\theta \end{matrix}\right\}$$

Non-uniqueness: for any $k \in \mathbb{Z}$

$$(r,\theta)_P = (r, \theta + 2k\pi)_P = (-r, \theta + (2k-1)\pi)_P$$

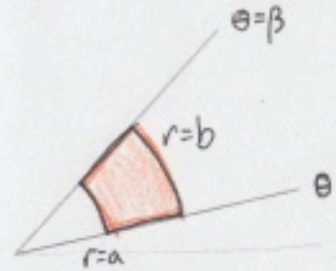
2

Definition: A polar rectangle is a Region $R \subset \mathbb{R}^2$ admitting a polar coordinate description

$$R = \{(r,\theta)_C \mid a \le r \le b,\ \alpha \le \theta \le \beta\}$$ for real #'s

$0 \le a < b$ and $\alpha < \beta$, $\beta - \alpha \in [0, 2\pi]$.

eg. The following pictures illustrate some polar rectangles



"The classic"

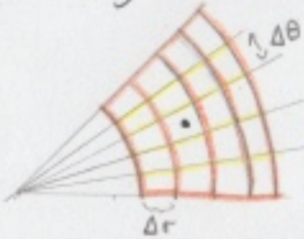
An annulus

3/4 disk / filled-Pacman-mid-bite

3.

For a polar rectangle R and f integrable over R, consider $\iint_R f(x,y)\,dA$. We wish to rewrite it as an iterated integral using polar coordinates, but we must determine how dA depend on r and θ. Intuitively, dA will not merely be $dr\,d\theta$, since for larger r, the area of a polar rectangle with constant Δr and $\Delta\theta$ is still larger:

polar rectangles with equal Δr and $\Delta\theta$, but unequal areas.



Subdividing R into R_{ij}'s

We subdivide R into sub-polar rectangles (polar subrectangles?) R_{ij}, eg. by partitioning the intervals $[a,b]$ and $[\alpha,\beta]$ uniformly (it doesn't matter if the spacing is non-uniform; this only introduces extra indices; in the limit the result is the same for integrable f.) The area of a polar subrectangle R_{ij} is a difference of Sector areas:

$$A(R_{ij}) = \frac{1}{2}\left[r_i^2(\theta_j - \theta_{j-1}) - r_{i-1}^2(\theta_j - \theta_{j-1})\right]$$

$$= \frac{1}{2}(r_i^2 - r_{i-1}^2)\Delta\theta$$

4.

$$= \left(\frac{r_i + r_{i-1}}{2}\right)(r_i - r_{i-1})\Delta\theta = r_i^* \Delta r \Delta\theta$$

where r_i^* is the average radius, and $(r_i^*, \theta_i^*)_P$ is the "center" of R_{ij}.

Then in the limit as we increase the # of subdivisions 5.

$$\Delta A = r_i^* \Delta r \Delta \theta \longrightarrow dA = r\,dr\,d\theta$$

Proposition: If $f(x,y)$ is integrable over a polar rectangle $R = \{(r,\theta)_p \mid a \le r \le b,\ \alpha \le \theta \le \beta\}$, then

$$\iint_R f(x,y)\,dA = \int_\alpha^\beta \int_a^b f(r\cos\theta, r\sin\theta)\, r\,dr\,d\theta.$$

Returning to the problem of verifying a radius R sphere's volume, we will convert the integral

$$\int_{-R}^{R} \int_{-\sqrt{R^2-x^2}}^{\sqrt{R^2-x^2}} 2\sqrt{R^2-x^2-y^2}\,dy\,dx$$

into polar coordinates, and compute it.

Observe that a radius R disk is a polar rectangle: 6.

$$D = \{(r,\theta)_p \mid 0 \le r \le R,\ 0 \le \theta \le 2\pi\}.$$

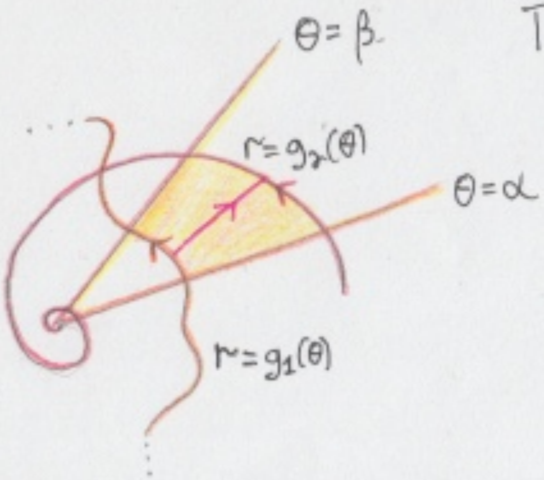
Thus the volume of a radius R sphere is

$$V = \iint_D f(x,y)\,dA = \int_0^{2\pi} \int_0^R 2\sqrt{R^2-r^2}\, r\,dr\,d\theta$$

$$= \left(\int_0^{2\pi} d\theta\right)\left(\int_0^R d\left(-\tfrac{2}{3}(R^2-r^2)^{3/2}\right)\right)$$

$$= \frac{4}{3}\pi R^3,$$ as expected.

Next we consider more general regions bounded by polar curves. Suppose $g_1(\theta)$ and $g_2(\theta)$ are functions such that $g_1(\theta) \le g_2(\theta)$ for $\theta \in [\alpha,\beta]$, and let $D = \{(r,\theta) \mid \alpha \le \theta \le \beta,\ g_1(\theta) \le r \le g_2(\theta)\}$. 7.



Then

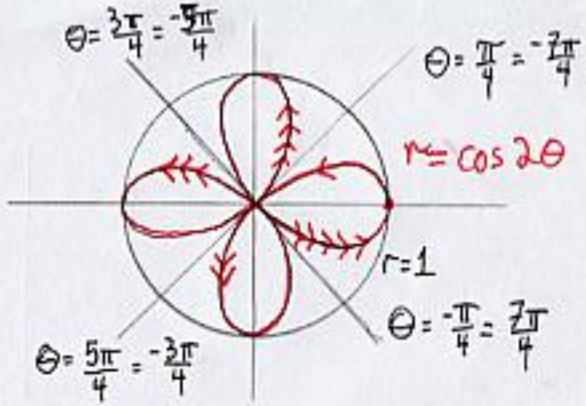
$$\iint_D f(x,y)\,dA \ \ \| \ \ \int_\alpha^\beta \int_{g_1(\theta)}^{g_2(\theta)} f(r\cos\theta, r\sin\theta)\, r\,dr\,d\theta$$

Observe that the preceding theorem recovers the usual formula for polar area bounded by $r = f(\theta)$, $\alpha \le \theta \le \beta$: 8.

$$A = \int_\alpha^\beta \int_0^{f(\theta)} r\,dr\,d\theta = \int_\alpha^\beta \tfrac{1}{2}[f(\theta)]^2\,d\theta.$$

We'll apply this in an example.

Ex: Let $r = \cos 2\theta$. Curves of the form $r = \cos(n\theta)$ are often called "roses" (More like daisies?).



Note: $r = \cos 2\theta = 0$ whenever $\theta = \frac{\pi}{4} + k\pi$, $k \in \mathbb{Z}$. Thus, it is tangent to rays $\theta = \frac{\pi}{4}, \frac{3\pi}{4}, \frac{5\pi}{4}, \frac{7\pi}{4}, \dots$ as it passes through $\vec{O}$. 9.

We will calculate the area of one petal, using that $\iint_P dA = A(P)$ where P is a petal, say, the one with $-\frac{\pi}{4} \le \theta \le \frac{\pi}{4}$.

The outer integral thus has bounds $-\frac{\pi}{4}$ to $\frac{\pi}{4}$, and the inner integral is bounded with inner radius $r=0$ and outer integral $r = \cos 2\theta$: 10.

$$A = \int_{-\frac{\pi}{4}}^{\frac{\pi}{4}} \int_0^{\cos 2\theta} r\,dr\,d\theta \underset{\text{by Symmetry across x-axis.}}{=} 2\int_0^{\frac{\pi}{4}} \int_0^{\cos 2\theta} r\,dr\,d\theta$$

Alternatively, appealing to the four-fold Symmetry:

$$A = \frac{1}{4}\int_0^{2\pi}\int_0^{\cos 2\theta} r\,dr\,d\theta.$$

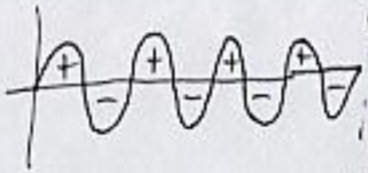
Evaluating via the latter integral: 11.

$$A = \frac{1}{4}\int_0^{2\pi} \frac{1}{2}[\cos 2\theta]^2\,d\theta = \frac{1}{8}\int_0^{2\pi} \cos^2 2\theta\,d\theta$$

$$= \frac{1}{8}\int_0^{2\pi} \frac{1+\cos 4\theta}{2}\,d\theta = \frac{\pi}{8} + \underbrace{\int_0^{2\pi} \frac{\cos 4\theta}{2}\,d\theta}_{=0}$$

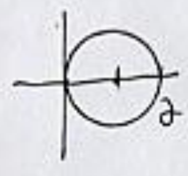
$$= \frac{\pi}{8}.$$

0, since the period of $\cos 4\theta$ is $\frac{\pi}{2}$, and $2\pi = 4(\frac{\pi}{2})$ is an integral multiple of this period.



Example: Find the Volume of the Solid beneath $z = x^2+y^2$ above the plane $z=0$, and inside the cylinder $x^2+y^2 = 2x$. 12.

Region: $r^2 = 2r\cos\theta \Rightarrow r = 2\cos\theta$, can choose $r>0$ so $-\frac{\pi}{2} \le \theta \le \frac{\pi}{2}$. This is a Circle of radius 1, centered at $(1,0)_p = (1,0)_e$



$$V = \iint_D x^2+y^2\,dA$$

$$= \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \int_0^{2\cos\theta} (r^2)\,r\,dr\,d\theta = \frac{3\pi}{2}.$$