CURVES AND MOTION VIA VECTOR VALUED FUNCTIONS

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0. Notes on these Notes

These notes cover material spanning several lectures, focusing on curves and functions whose outputs are viewed as position vectors. The notes also contain a few additional remarks on curves and examples that could not be covered in class, as well as some good exercises. A separate, optional, and more challenging set of notes has been made to illuminate the meanings of various geometric objects associated to curves, such as vector and scalar curvature, torsion, and natural frames: see *Curvature, Natural Frames, and Acceleration for Plane and Space Curves* as linked on the course website.

1. Vector Valued Functions and Curves

1.1. **Describing Curves Parametrically.** One often first encounters plane curves as graphs of continuous, single variable real-valued functions, and subsequently is introduced to implicit curves. For example, you are probably familiar with the '$\cup$' shape of the graphs of functions $y = x^{2n}$, $n = 1, 2\ldots$, and you have surely encountered the *unit circle* as the locus of all points $(x, y) \in \mathbb{R}^2$ satisfying $x^2 + y^2 = 1$, which is likely the first implicitly defined curve to become familiar to you.

One can also take another approach: introduce a parameter, and describe each of the constituent entries of the ordered pair (x, y) as functions of this common parameter, resulting in a *parametric description of a curve*. For example, the unit circle can be described by many parameterizations, but perhaps the simplest descriptions are

$$x(\theta) = \cos\theta\,,\ y(\theta) = \sin\theta\,,\ 0 \le \theta \le 2\pi\,, \qquad \text{or}$$

$$x(t) = \sin(t)\,,\ y(t) = \cos(t)\,,\ t \in \mathbb{R}\,.$$

Exercise 1.1. Describe the differences in these parameterizations from a geometric standpoint (what do θ and t represent?). Then, imagine that the parameterizations describe particle motion, and explain the differences in their motions.

One can recover curves which are graphs from a parameterization as well: simply let $x(t) = t$, and $y(t) = f(t)$, and the resulting curve, for $t \in \mathrm{Dom}(f)$ is simply the graph $y = f(x)$. This parameterization is of course not unique: we can replace t by any function $g(t)$ which is monotonic, and whose range contains $\mathrm{Dom}(f)$ and obtain a new parameterization. For example, if $\mathrm{Dom}(f) = \mathbb{R}$ one could also parameterize the graph of $y = f(x)$ by $x(t) = t^3$, $y(t) = f(t^3)$. Viewing the parameterization as describing *motion* along the curve, we realize that these parameterizations are quite different in nature, though they describe the same plane curve.

Parameterizations have greater freedom to describe curves than graphs, or even implicit functions. As a testament to this, consider the cycloid: a cycloid is the shape made by following a point on the circumference of a disk, as the disk rolls without slipping along a straight path. If the disk has radius R and rolls along the x-axis starting at time $t = 0$, one can deduce that the cycloid has parametric equations

$$x(t) = R(t - \sin t) \quad y(t) = R(1 - \cos t) \quad 0 \le t\,.$$

One can express a portion of one arch by an equation giving x as a function of y, but no closed form f exists for which $y = f(x)$ describes the whole cycloid.

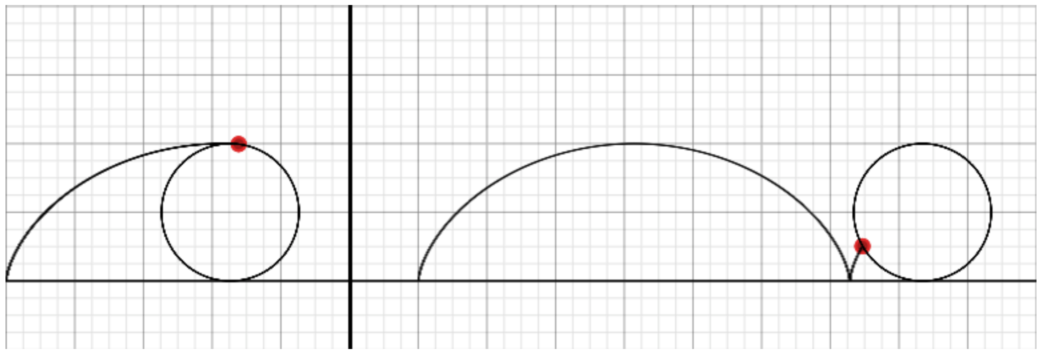


FIGURE 1. A pair of moments in a disk's motion, following one point on the circumference as it traces out a cycloid.

Exercise 1.2. Use a diagram and basic trigonometric considerations to demonstrate that the above parameterization of the cycloid is correct. Describe a geometric reinterpretation of t (not as merely time), and then, presuming t *is* time in seconds, determine the linear horizontal speed of the rolling disk's center. Presume that x, y and R are expressed in meters.

In three dimensions, one can describe curves in space parametrically by specifying a parameter and three functions defined on some common domain (which may be a proper subset of the natural domains of any of the given functions). That is, one may describe a curve $\mathcal{C}$ to be the locus of points determined parametrically by the triple of equations

$$x(t) = f(t)\,,\quad y(t) = g(t)\,,\quad z(t) = h(t)\,,\quad t \in D \subseteq \mathbb{R}\,.$$

The curve $\mathcal{C}$ is connected if and only if D is a connected subset of $\mathbb{R}$ and all three functions are continuous on any open subinterval of D.

We've already encountered a simple example: parametric equations to describe lines in $\mathbb{R}^3$. In particular, if f, g, and h, are all linear functions of t, then the corresponding parametric equations describe a line if $D = \mathbb{R}$ and a line segment if $D = [a, b]$ is an interval. We already know that we can organize this same data using a vector whose components are the three functions f, g, and h.

Exercise 1.3. : Give a geometric argument that the line segment from (x_0, y_0, z_0) to (x_1, y_1, z_1) may be parameterized by

$$x(t) = (1-t)x_0 + tx_1\,,\quad y(t) = (1-t)y_0 + ty_1\,,\quad z(t) = (1-t)z_0 + tz_1\,,\quad t \in [0,1]\,.$$

Give a new parameterization for a time parameter t (in seconds) that describes a particle smoothly traveling from (x_0, y_0, z_0) to (x_1, y_1, z_1) and back to (x_0, y_0, z_0) in as many seconds as the twice length of the line segment.

1.2. **Vector Valued Functions.**

Definition. Given a domain $D \subset \mathbb{R}$, a (3D) vector valued function is a vector valued map $\mathbf{r} : D \to \mathbb{R}^3$, described by specifying functions $f, g, h : D \to \mathbb{R}$:

$$\mathbf{r}(t) = f(t)\,\hat{\imath} + g(t)\,\hat{\jmath} + h(t)\,\hat{\mathbf{k}} = \langle f(t), g(t), h(t)\rangle\,, t \in D\,.$$

One can specify a vector valued function with values in $\mathbb{R}^2$ by omitting the third component; more generally one can define vector valued functions with n-dimensional vector values by choosing n functions $f_1, \ldots, f_n : D \to \mathbb{R}$ as the components of an n-vector $\mathbf{r}(t) = \langle f_1(t), \ldots, f_n(t)\rangle$. The *domain of* $\mathbf{r}$, D, is often denoted $\mathrm{Dom}(\mathbf{r})$. The *image* or *range* of $\mathbf{r}$ is the set of points in $\mathbb{R}^3$ (or $\mathbb{R}^2$, or $\mathbb{R}^n$) whose position vectors are values of $\mathbf{r}(t)$ for some t.

Note that the components must all be defined on a common domain. One can ask about *natural domain*, as one does for single variable functions: for a given $\mathbf{r}$ described by component formulae, what is the largest domain in $\mathbb{R}$ on which $\mathbf{r}$ is defined?

Proposition. *The natural domain of a vector valued function is the intersection of the natural domains of the components.*

Exercise 1.4. : Prove the above proposition.

A vector valued function is equivalent to parametric equations, but provides us a neat geometric language to work with curves. Indeed, if the domain is connected and the component functions have (locally) connected graphs (meaning no jumps/skips or undefined points), then the image of a two or three dimensional vector valued function is a curve in $\mathbb{R}^2$ or $\mathbb{R}^3$ respectively.

Example. Describe and visualize the curve given parametrically by $\mathbf{r}(t) = \cos(t)\,\hat{\imath} + \sin(t)\,\hat{\jmath} + t\,\hat{\mathbf{k}}$.

Solution: First, consider the "shadow" (projection) of this parameterization on the xy-plane. Clearly, this is a unit circle, traversed counterclockwise. Indeed, one can easily verify that the x and y components of $\mathbf{r}$ satisfy $x^2 + y^2 = 1$. Thus, the whole curve is in fact confined to this cylinder. The z-component tells the rest of the story: as t increases, the height above the xy-plane increases commensurately. Thus, this curve is a *circular helix*.

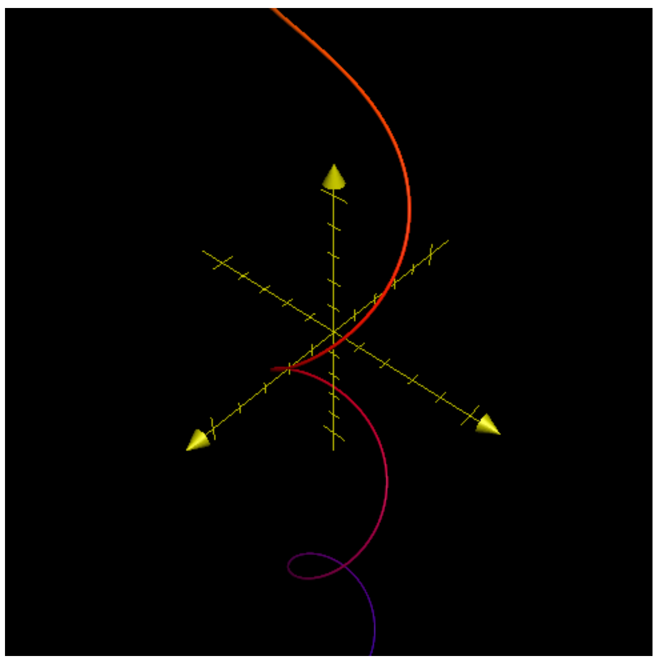


FIGURE 2. The circular helix parameterized by $\mathbf{r}(t) = \cos(t)\,\hat{\imath} + \sin(t)\,\hat{\jmath} + t\,\hat{\mathbf{k}}$.

Exercise 1.5. Recall that conic sections have cartesian equations of the following form:

$$\text{Ellipse: } \frac{x^2}{a^2} + \frac{y^2}{b^2} = 1\,, \quad \text{Parabola: } y = ax^2\,, \quad \text{Hyperbola: } \frac{x^2}{a^2} - \frac{y^2}{b^2} = \pm 1\,.$$

For each, give a vector valued function parameterizing the curves with a domain of $\mathbb{R}$.

Exercise 1.6. Give Cartesian equations for the following plane curves:

(a.) The curve which is the image of the vector valued function $\mathbf{r}(t) = t^2\hat{\imath} + t^4\hat{\jmath}$, $t \in \mathbb{R}$.

(b.) The trajectory of a particle whose position after t seconds is $\mathbf{r}(t) = \langle \sin^3(\pi t), 1 - \sin^2(\pi t)\rangle$, $0 \le t \le 1$.

(c.) The "witch of Agnesi," a curve named for Maria Agnesi, author of the first three-term calculus text, given by the vector valued function $\mathbf{r}(t) = 2a\cot(\theta)\hat{\imath} + 2a\sin^2(\theta)\hat{\jmath}$, $t \in [0, 2\pi]$, where a is a positive real constant.

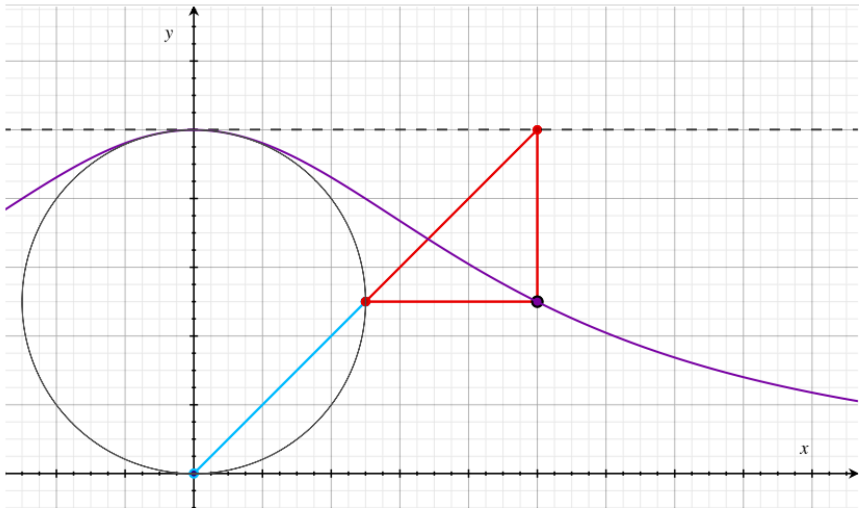


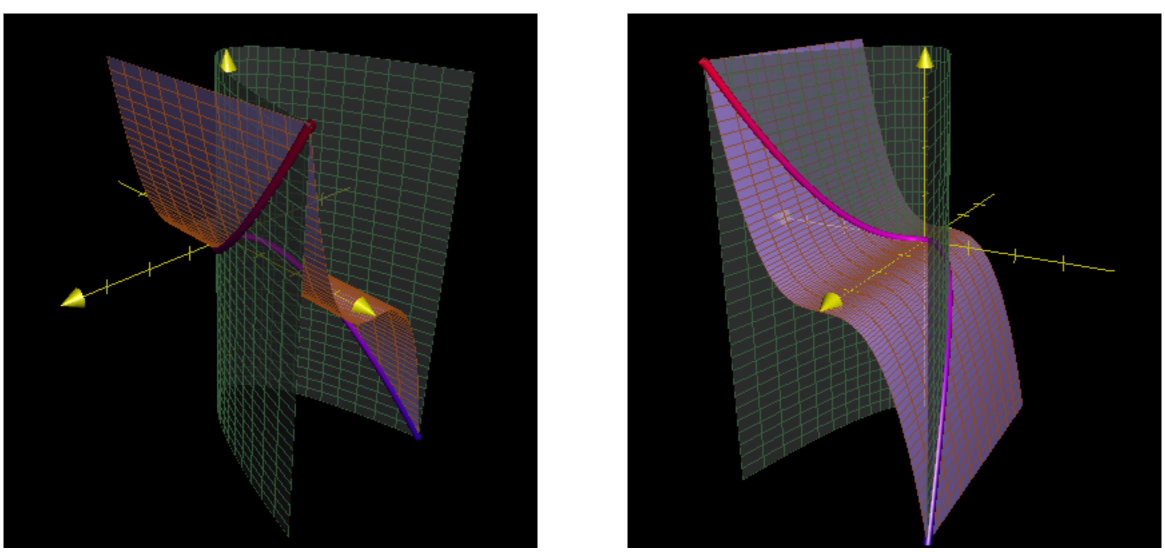
FIGURE 3. The Witch of Agnesi, "witch" being a mistranslation of the Italian word *versiera* (perhaps for the 'versine' function $1 - \cos t$.)

Whereas in the plane, a single equation involving both variables may implicitly specify a curve (such as a conic section), curves in 3-space often arise as the intersection of a pair of surfaces. Indeed, conic sections are realized as intersections of a double cone with a plane, hence their name. But non-planar curves arise, e.g. as the intersection loci of cylinders or other surfaces.

Example. Sketch the curve of intersection of the parabolic cylinder $y = x^2$ and the cubic cylinder $z = x^3$, and parameterize this curve.

Solution: $y = x^2$ gives a parabolic cylinder whose traces look like the familiar parabola translated into planes of constant z, while the cubic cylinder appears as the standard cubic sketched in the xz-plane, then translated into planes of constant y. They intersect along a path with twists upwards through the origin, from the octant where $x, z \leq 0$, $y > 0$ to the octant where x, y, and z are all positive. Letting $x = t$, we obtain a parameterization

$$\mathbf{r}(t) = t\hat{\mathbf{\imath}} + t^2\hat{\mathbf{\jmath}} + t^3\hat{\mathbf{k}}\,, t \in \mathbb{R}\,.$$



(A) (B)

FIGURE 4. Two views of the twisted cubic curve, realized as the intersection of a cubic cylinder with a parabolic cylinder.

Exercise 1.7. Describe the curves of intersection of the two cylinders $x^2+y^2=1$ and $y^2+z^2=1$ by a pair of vector valued functions, and sketch the cylinders showing these intersection curves. Use algebra to demonstrate that these curves are ellipses.

Exercise 1.8. Give a vector valued function which parameterizes a circle with center $C(1,3,-2)$ and contains the point $P(2,4,0)$, in the plane perpendicular to $\hat{\mathbf{\imath}}+\hat{\mathbf{\jmath}}+\hat{\mathbf{k}}$ containing both C and P.

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2. Calculus of Vector Valued Functions

Throughout this section, we'll define objects component-wise, presuming three dimensional vector valued functions unless otherwise indicated.

2.1. **Limits.** We will want to be able to describe limits, derivatives, and integrals of vector valued functions. Fortunately, we can work component-wise in each case.

Definition. The limit of the vector valued function $\mathbf{r}(t)=f(t)\,\hat{\mathbf{\imath}}+g(t)\,\hat{\mathbf{\jmath}}+h(t)\,\hat{\mathbf{k}}$ as the variable parameter t approaches a value a, is

$$\lim_{t\to a}\mathbf{r}(t):=\left(\lim_{t\to a}f(t)\right)\,\hat{\mathbf{\imath}}+\left(\lim_{t\to a}g(t)\right)\,\hat{\mathbf{\jmath}}+\left(\lim_{t\to a}h(t)\right)\,\hat{\mathbf{k}}\,,$$

provided each of the component limits exists.

It is an easy exercise to see that limit properties such as linearity hold in the context of limits of vector valued functions. One can also define continuity as one might hope:

Definition. The vector valued function $\mathbf{r}:D\to\mathbb{R}$ is *continuous at* $a\in D$ if and only if

$$\lim_{t\to a}\mathbf{r}(t)=\mathbf{r}(a)\,.$$

The vector valued function $\mathbf{r}$ is said to be continuous on a set $S\subseteq D$ if it is *continuous* at $t=a$ for each $a\in S$. If $\mathbf{r}$ is continuous on its domain D then we simply say "$\mathbf{r}$ is continuous."

Proposition. *A vector valued function* $\mathbf{r}(t)$ *is continuous at* $t=a$ *if and only if each of its component functions is continuous at* a.

Example. Compute the limit of $\mathbf{r}(t)=\left\langle te^t,\frac{\sin t}{t},3-e^{-t}\cos t\right\rangle$ as $t\to 0$. What is the natural domain of $\mathbf{r}(t)$?

Solution:

$$\lim_{t\to 0}\mathbf{r}(t)=\left\langle \lim_{t\to 0}(te^t),\lim_{t\to 0}\left(\tfrac{\sin t}{t}\right),\lim_{t\to 0}(3-e^{-t}\cos t)\right\rangle=\langle 0,1,2\rangle\,.$$

The natural domain of $\mathbf{r}(t)$ is $\mathbb{R}-\{0\}$, since the x and z components are defined for all real numbers, while the y component is undefined at 0, and otherwise defined. Observe that the function is in fact continuous at all points of its domain, and may be modified to continuous function by replacing the y component with its continuous analogue (which equals 1 at $t=0$, and elsewhere agrees with $(\sin t)/t$).

Exercise 2.1. Compute the limit

$$\lim_{t\to-\infty}\left[t^2e^t\hat{\mathbf{\imath}}-\arctan(t^{1/3})\,\hat{\mathbf{\jmath}}+\tfrac{\sqrt{9t^6-t^4+1}}{2t^3+7}\hat{\mathbf{k}}\right]\,.$$

Exercise 2.2. Find the natural domains of the following vector valued functions, then sketch the corresponding curves in either the plane $\mathbb{R}^2$ or in space $\mathbb{R}^3$ as appropriate.

(a.) $\mathbf{u}(t)=\cos\Big(\ln(1-t^2)\Big)\hat{\mathbf{\imath}}+\sin\Big(\ln(1-t^2)\Big)\hat{\mathbf{\jmath}}+\sin\Big(\ln[(1-t^2)^2]\Big)\hat{\mathbf{k}}$,

(b.) $t\mapsto t^{1/3}\langle (t^{-2/3}-1)^{1/2},1\rangle$,

(c.) $\mathbf{r}(t)=\sin(t^{-1})\,\hat{\mathbf{\imath}}+\cos(t^{-1})\,\hat{\mathbf{\jmath}}+\tan^{-1}\sqrt{t}\,\hat{\mathbf{k}}$.

2.2. **Differentiation of Vector Valued Functions.** Since vector valued functions give a natural way to describe plane and space curves, we wish to be able to compute tangent lines, and understand how the choice of parameterization describes motion. Thus we must consider differentiation of vector valued functions with respect to the parameter. As with limits, we will obtain a definition which allows us to compute derivatives component-wise.

Definition. The derivative of the vector valued function $\mathbf{r}(t)$ at $t = t_0$ is

$$\frac{\mathrm{d}\mathbf{r}}{\mathrm{d}t}\Big|_{t=t_0} = \dot{\mathbf{r}}(t_0) = \lim_{t\to t_0} \frac{\mathbf{r}(t) - \mathbf{r}(t_0)}{t - t_0}\,,$$

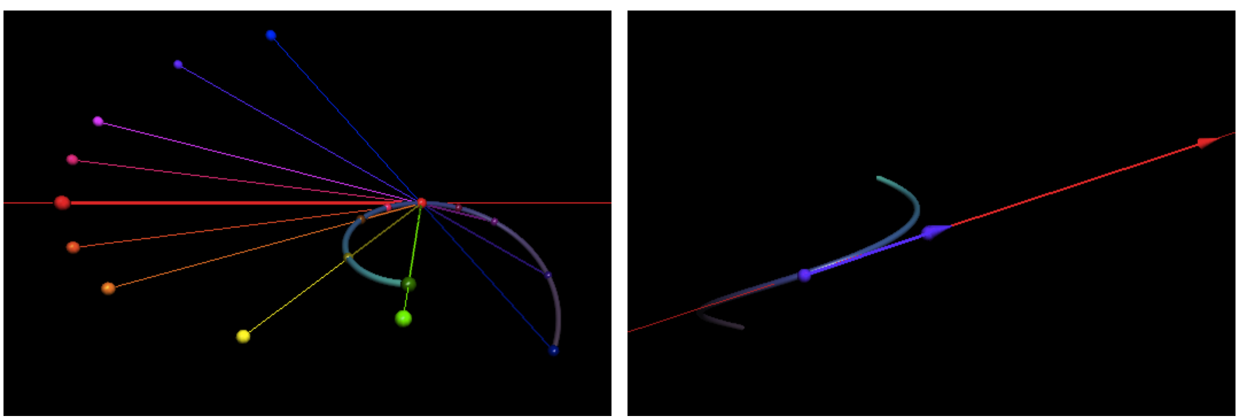
provided the limit exists. Here, the notation $\dot{\mathbf{r}}$ is an alternative to the notation $\mathbf{r}'$. The dot notation was introduced by Isaac Newton and is sometimes called Newton's fluxion notation. In physics, it is often reserved to denote derivative with respect to a time parameter, but we may use it generally to mean the derivative with respect to a parameter labeled t.

Proposition. *The limit* $\lim_{t\to t_0} \frac{\mathbf{r}(t)-\mathbf{r}(t_0)}{t-t_0}$ *exists if and only if each component of* $\mathbf{r}$ *is differentiable at* $t = t_0$*, in which case*

$$\dot{\mathbf{r}}(t_0) = \dot{x}(t_0)\hat{\imath} + \dot{y}(t_0)\hat{\jmath} + \dot{z}(t_0)\hat{\mathbf{k}}\,.$$

One can then naturally consider the derivative function $\dot{\mathbf{r}}(t) = \dot{x}(t)\hat{\imath} + \dot{y}(t)\hat{\jmath} + \dot{z}(t)\hat{\mathbf{k}}$, which also describes a space curve. The geometric interpretation of its value at a point is as a *tangent vector*: $\dot{\mathbf{r}}(t_0)$ gives a vector tangent to the curve $\mathbf{r}(t)$ at the position $\mathbf{r}(t_0)$, and is called the *velocity vector of the parameterized curve* $\mathbf{r}(t)$ *at time* $t = t_0$.

To see that this is so, consider first that the direction of $\dot{\mathbf{r}}(t_0)$ is the limit of *secant directions*: the difference quotient is formed by taking a displacement $\mathbf{r}(t) - \mathbf{r}(t_0)$ emanating from $\mathbf{r}(t_0)$ and terminating in a nearby point $\mathbf{r}(t)$, then scaling by $1/(t-t_0)$. In the limit, as the direction approaches the tangential direction of motion, the magnitude simultaneously approaches the instantaneous speed for the parametrization. If $\dot{\mathbf{r}}(t) \neq 0$, then we can capture information intrinsic to the curve (and not dependent on regarding it as a particular trajectory) via a *unit tangent vector*



(A) (B)

FIGURE 5. (A): secant vectors and difference quotient vectors, with a limiting tangent vector $\dot{\mathbf{r}}(t_0)$ and tangent line (in red), illustrated for the helix $\mathbf{r}(t) = \cos(3t)\hat{\imath} + \sin(3t)\hat{\jmath} + t\hat{\mathbf{k}}$ with $t_0 = 0$. (B): the tangent vector $\dot{\mathbf{r}}(0)$ (in red) and the corresponding unit tangent vector $\mathbf{T}(0)$ (in purple) for this helix.

Definition. The unit tangent vector $\mathbf{T}$ to a curve given parametrically by $\mathbf{r}(t) = x(t)\hat{\imath}+y(t)\hat{\jmath}+z(t)\hat{\mathbf{k}}$ is

$$\mathbf{T}(t_0) = \frac{\dot{\mathbf{r}}(t_0)}{\|\dot{\mathbf{r}}(t_0)\|}\,;$$

it is well defined at t_0 provided $\dot{\mathbf{r}}(t_0)$ exists and is nonzero.

If $\mathbf{r}(t)$ is a parameterization such that $\dot{\mathbf{r}}(t_0)$ exists and is nonzero for all t, we say that $\mathbf{r}$ is *regular*. Observe then that for a regular curve, the unit tangent vector exists and is well defined for all t. If $\mathbf{r}(t)$ is parameterized so that $\|\dot{\mathbf{r}}(t)\| = 1$ for all t, so that $\dot{\mathbf{r}} = \mathbf{T}$, we call it a *unit speed parameterization*. Unit speed parameterizations have the nice property that, up to a constant, they parameterize the curve by distance traveled along it (see the section on arc-length.)

The following theorem gives the fundamental properties of differentiation for vector valued functions.

Theorem. *Let $\mathbf{u}(t)$ and $\mathbf{v}(t)$ be differentiable vector valued functions of the same dimension, let $g(t)$ be a real valued differentiable function, and let c be a constant scalar. Then*

(1) $\dfrac{\mathrm{d}}{\mathrm{d}t}\big(\mathbf{u}(t) + \mathbf{v}(t)\big) = \dot{\mathbf{u}}(t) + \dot{\mathbf{v}}(t)\,,$

(2) $\dfrac{\mathrm{d}}{\mathrm{d}t}\big(c\mathbf{u}(t)\big) = c\dot{\mathbf{u}}(t)\,,$

(3) $\dfrac{\mathrm{d}}{\mathrm{d}t}\big(g(t)\mathbf{u}(t)\big) = \dot{g}(t)\mathbf{u}(t) + g(t)\dot{\mathbf{u}}(t)\,,$

(4) $\dfrac{\mathrm{d}}{\mathrm{d}t}\big(\mathbf{u}(t)\cdot\mathbf{v}(t)\big) = \dot{\mathbf{u}}(t)\cdot\mathbf{v}(t) + \mathbf{u}(t)\cdot\dot{\mathbf{v}}(t)\,,$

(5) $\dfrac{\mathrm{d}}{\mathrm{d}t}\big(\mathbf{u}(t)\times\mathbf{v}(t)\big) = \dot{\mathbf{u}}(t)\times\mathbf{v}(t) + \mathbf{u}(t)\times\dot{\mathbf{v}}(t)\,,$

(6) $\dfrac{\mathrm{d}}{\mathrm{d}t}\Big(\mathbf{u}\big(g(t)\big)\Big) = \dot{g}(t)\dot{\mathbf{u}}\big(f(t)\big)\,.$

Exercise 2.3. Prove the above theorem.

Example. Suppose $\mathbf{r}(t)$ is a twice differentiable vector valued function such that there is a constant c with $\|\dot{\mathbf{r}}(t)\| = c$ for all $t \in \mathrm{Dom}(\mathbf{r})$. Then the first and second derivatives of $\mathbf{r}$ are always orthogonal: $\dot{\mathbf{r}}(t)\cdot\ddot{\mathbf{r}}(t) = 0$ for all $t \in \mathrm{Dom}(\mathbf{r})$.

Indeed, by property (4) in the theorem above, $\frac{\mathrm{d}}{\mathrm{d}t}\big(\dot{\mathbf{r}}(t)\cdot\dot{\mathbf{r}}(t)\big) = \ddot{\mathbf{r}}(t)\cdot\dot{\mathbf{r}}(t) + \dot{\mathbf{r}}(t)\cdot\ddot{\mathbf{r}}(t) = 2\dot{\mathbf{r}}(t)\cdot\ddot{\mathbf{r}}(t)$. But $\dot{\mathbf{r}}(t)\cdot\dot{\mathbf{r}}(t) = c^2$ for all t, and $\frac{\mathrm{d}}{\mathrm{d}t}(c^2) = 0$. Thus, $\dot{\mathbf{r}}(t)\cdot\ddot{\mathbf{r}}(t) = 0$ for all $t \in \mathrm{Dom}(\mathbf{r})$, and so $\dot{\mathbf{r}}(t) \perp \ddot{\mathbf{r}}(t)$. Geometrically, one sees that the constant speed condition implies that the *velocity curve* $\dot{\mathbf{r}}(t)$ lies on a sphere of radius c, and the second derivative $\ddot{\mathbf{r}}$ describes the bending of this curve, which is necessarily tangent to the sphere. But then, since $\dot{\mathbf{r}}(t)$ is a position vector on the sphere, its tangential direction $\ddot{\mathbf{r}}(t)$ is perpendicular to it, as tangents to spheres are perpendicular to radii.

Proposition. *If $\mathbf{r}(t)$ is differentiable at t_0 with derivative $\dot{\mathbf{r}}(t_0)$, then the tangent line to the curve parameterized by $\mathbf{r}(t)$, with point of tangency positioned at $\mathbf{r}(t_0)$, may be parameterized by*

$$\boldsymbol{\ell}(s) = \mathbf{r}(t_0) + s\dot{\mathbf{r}}(t_0)\,.$$

Example. Describe the curve $\mathcal{C}$ of intersection of quadric $z = xy$ with the cylinder $x^2 + y^2 = 4$ parametrically, and give the equation of a tangent line to the curve at the point $(\sqrt{2}, \sqrt{2}, 2)$.

Solution: Observe that since one of the surfaces is $x^2 + y^2 = 4$, we must have that the x and y component functions to satisfy this equation. Such functions parameterize a radius two circle in the plane. We may take, for simplicity, $x(t) = 2\cos t$ and $y(t) = 2\sin t$. Then along the curve of intersection, $z(t) = x(t)y(t) = 4\cos t\sin t = 2\sin 2t$. Thus, let $\mathbf{r}(t) = 2\Big(\cos(t)\,\hat{\imath} + \sin(t)\,\hat{\jmath} + \sin(2t)\,\hat{\mathbf{k}}\Big)$.

For the tangent line, we compute a value t_0 of the parameter t giving us the point $(\sqrt{2}, \sqrt{2}, 2)$ by equating like components. This gives the equations $\sqrt{2} = \cos t = \sin t$, $2 = \sin 2t$. Observe that taking $t_k = \pi/4 + 2k\pi$ (for any integer k) simultaneously satisfies the conditions; we may take $k = 0$

giving $t_0 = \pi/4$. Then we compute the tangent vector

$$\begin{aligned}\dot{\mathbf{r}}(\pi/4) &= \frac{\mathrm{d}}{\mathrm{d}t}\langle 2\cos t, 2\sin t, 2\sin 2t\rangle\Big|_{t=\pi/4}\\ &= \langle -2\sin t, 2\cos t, 4\cos 2t\rangle\Big|_{t=\pi/4}\\ &= \langle -\sqrt{2}, \sqrt{2}, 0\rangle\end{aligned}$$

Thus, $\boldsymbol{\ell}(s) = \langle\sqrt{2}, \sqrt{2}, 2\rangle + s\langle -\sqrt{2}, \sqrt{2}, 0\rangle = \langle\sqrt{2}(1-s), \sqrt{2}(1+s), 2\rangle$.

Caution: I've observed that some students when seeking a tangent line compute the derivative as a function of t and forget to evaluate at some value $t = t_0$ to obtain a *constant tangent vector*. This can result in students giving parametric equations for something other than a line as an answer to a tangent line question. Be sure to do a reality check: is the answer really *linear* in its parameter?

Example. Compute the unit tangent $\mathbf{T}$ vector of the helix $\mathbf{r}(t) = \langle\cos t, \sin t, t\rangle$ as a function of t. What kind of curve does $\mathbf{T}(t)$ trace out on the unit sphere?

Solution: The velocity vector of the helix is $\dot{\mathbf{r}}(t) = \langle -\sin t, \cos t, 1\rangle$, whence

$$\mathbf{T}(t) = \frac{\dot{\mathbf{r}}(t)}{\|\dot{\mathbf{r}}(t)\|} = \frac{\langle -\sin t, \cos t, 1\rangle}{\sqrt{\sin^2 t + \cos^2 t + 1}} = \frac{1}{\sqrt{2}}\langle -\sin t, \cos t, 1\rangle\,.$$

Observe that on the unit sphere, this traces out a circle of radius $\sqrt{2}/2$, given by the intersection of the unit sphere with the plane $z = \sqrt{2}/2$.

Exercise 2.4. Compute the following derivatives:

(a.) $\dfrac{\mathrm{d}}{\mathrm{d}t}\Big(\sec t\,\hat{\mathbf{\imath}} + \csc t\,\hat{\mathbf{\jmath}} + t/(\sqrt{1-t^2})\hat{\mathbf{k}}\Big)$,

(b.) $\dfrac{\mathrm{d}}{\mathrm{d}t}\Big[e^{-t^2}(\cos t^2\,\hat{\mathbf{\imath}} - \sin t^3\,\hat{\mathbf{\jmath}})\Big]$,

(c.) $\dfrac{\mathrm{d}}{\mathrm{d}t}\left[\mathbf{a}t \times (\mathbf{v}t + \mathbf{b})\right]$, where $\mathbf{a}$, $\mathbf{b}$, and $\mathbf{v}$ are constant vectors,

(d.) $\dfrac{\mathrm{d}}{\mathrm{d}t}\left[\mathbf{a}t \cdot (\mathbf{u}t + \mathbf{b}) \times (\mathbf{v}t + \mathbf{c})\right]$, where $\mathbf{a}$, $\mathbf{b}$, $\mathbf{c}$, $\mathbf{u}$ and $\mathbf{v}$ are constant vectors.

Exercise 2.5. Find symmetric equations of the tangent line to the point $P(2, 8, 32)$ on the curve $\mathbf{r}(t) = \langle e^t, e^{3t}, e^{5t}\rangle$.

Example. A parametric curve $\mathbf{r}(t)$ can fail to be regular at a point $\mathbf{r}(t_0)$ if $\dot{\mathbf{r}}(t_0) = \mathbf{0}$. Since the speed must go to zero at such a point, no unit tangent vector can be defined there from the given parameterization. But this does not mean one cannot exist for another parameterization. However, if the curve itself is not *smooth* there, then no such parameterization exists. Consider for example, the *cusp* at $(0,0)$ of the plane curve $y^2 = x^3$. This may be parameterized by $\mathbf{r}(t) = \langle t^2, t^3\rangle$. The velocity function for this parameterization is then $\dot{\mathbf{r}}(t) = \langle 2t, 3t^2\rangle$, which clearly vanishes at $t = 0$. A particle traversing this curve according to $\mathbf{r}(t)$ will slow down, stop, and change direction at the origin, before curving upwards. Indeed

$$\begin{aligned}\lim_{t\to 0^-}\frac{\dot{\mathbf{r}}(t)}{\|\dot{\mathbf{r}}(t)\|} &= \left\langle \lim_{t\to 0^-}\frac{2t}{\sqrt{4t^2+9t^4}}, \lim_{t\to 0^-}\frac{3t^2}{\sqrt{4t^2+9t^4}}\right\rangle = \langle -1, 0\rangle\,,\\ \lim_{t\to 0^+}\frac{\dot{\mathbf{r}}(t)}{\|\dot{\mathbf{r}}(t)\|} &= \left\langle \lim_{t\to 0^+}\frac{2t}{\sqrt{4t^2+9t^4}}, \lim_{t\to 0^+}\frac{3t^2}{\sqrt{4t^2+9t^4}}\right\rangle = \langle 1, 0\rangle\,.\end{aligned}$$

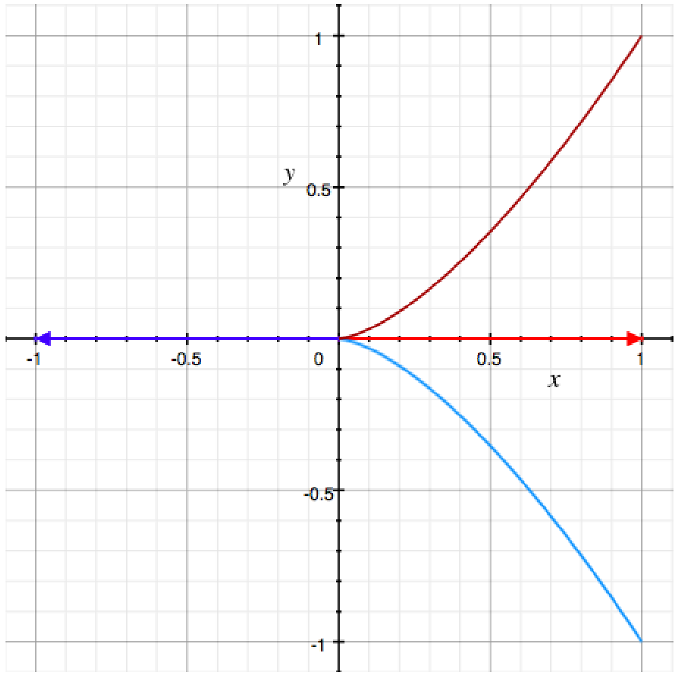


FIGURE 6. The cusp of the curve $\mathbf{r}(t) = \langle t^2, t^3 \rangle$, $t \in [-1, 1]$ and the limiting unit tangent vectors at the cusp. The blue portion of the curve corresponds to negative t, and the red to positive t.

2.3. **Integration of Vector Valued Functions.** We may define the Riemann integral of a vector valued function as one might define an integral of a single variable function: first we partition the domain of integration, and then we form an appropriate Riemann sum. While the details of such a formal definition may be worked out below in the exercises, it should be obvious that the result is expressible as a component-wise integral:

$$\int_a^b \mathbf{r}(t)\,\mathrm{d}t = \left(\int_a^b f(t)\,\mathrm{d}t\right)\hat{\imath} + \left(\int_a^b g(t)\,\mathrm{d}t\right)\hat{\jmath} + \left(\int_a^b h(t)\,\mathrm{d}t\right)\hat{\mathbf{k}}\,.$$

We may also easily obtain a vector-valued version of the fundamental theorem of calculus:

Theorem. *If $\mathbf{R} : [a,b] \to \mathbb{R}^3$ is a vector valued function such that $\dot{\mathbf{R}}(t) = \mathbf{r}(t)$ for all $t \in (a,b)$, and if $\mathbf{r}(t)$ is Riemann integrable over $[a,b]$, then*

$$\int_a^b \mathbf{r}(t)\,\mathrm{d}t = \mathbf{R}\Big|_a^b = \mathbf{R}(b) - \mathbf{R}(a)\,.$$

Further, for any $\mathbf{r} : [a,b] \to \mathbb{R}^3$ continuous except for isolated jump discontinuities, the function $\mathbf{R}(t) = \int_a^t \mathbf{r}(\tau)\,\mathrm{d}\tau$ furnishes such an antiderivative. In particular, such $\mathbf{R}$ is continuous on $[a,b]$ and differentiable on (a,b) with velocity function $\dot{\mathbf{R}}(t) = \mathbf{r}(t)$.

Exercise 2.6. This exercise outlines the scheme by which one may define vector valued integration.

(a.) Recall the definition of a partition of an interval $[a,b]$ (if you forget, you should reference a decent calculus textbook, where rectangular Riemann sums are first introduced). Write down what it means to have a uniform partition with n subintervals. Recall, the *norm of the partition* is the maximum length of a subinterval. What's the norm of the uniform partition of $[a,b]$ into n subintervals?

(b.) Mirroring the definition of a Riemann sum of a single variable function $f : [a,b] \to \mathbb{R}$, write down a definition of a Riemann sum for a vector-valued function, first without using components. Whereas the terms of a Riemann sum for $f(t)$ are geometrically realized by signed rectangle areas, what is the geometric meaning of the terms of a vector-valued Riemann sum?

(c.) Write down a careful definition of the Riemann integral $\int_a^b \mathbf{r}(t)\,\mathrm{d}t$ as a limit of Riemann sums.

Example. Compute the integral of $\mathbf{r}(t) = \sin^3 t\hat{\mathbf{\imath}} + (t^2 - t + t^{-1} + t^{-2/3})\hat{\mathbf{\jmath}} + \frac{e^{\sqrt{t}}}{\sqrt{t}}\hat{\mathbf{k}}$ for $\pi/2 \le t \le \pi$.

Solution: We have to integrate each component. For the $\hat{\mathbf{\imath}}$-component, we can use the Pythagorean identity to rewrite $\sin^3 t$ as $\sin t - \cos^2 t \sin t$, and then use substitution for the latter term. The $\hat{\mathbf{\jmath}}$-component can be integrated using the power rule, remembering that $\ln t$ is the antiderivative for t^{-1}. The final component may be treated using the substitution $u = \sqrt{t}$, or one can recognize that twice the derivative of $\sqrt{t}$ shows up in the integrand, and then undo the chain rule directly. Thus

$$\begin{aligned}\int_{\pi/2}^{\pi} \mathbf{r}(t)\,\mathrm{d}t &= \left\langle \int_{\pi/2}^{\pi} \sin^3 t\,\mathrm{d}t, \int_{\pi/2}^{\pi} t^2 - t + t^{-1} + t^{-2/3}\,\mathrm{d}t, \int_{\pi/2}^{\pi} \frac{e^{\sqrt{t}}}{\sqrt{t}}\,\mathrm{d}t \right\rangle \\ &= \left\langle -\cos t + \frac{1}{3}\cos^3 t\Big|_{\pi/2}^{\pi}, \frac{t^3}{3} - \frac{t^2}{2} + \ln t + 3t^{1/3}\Big|_{\pi/2}^{\pi}, 2e^{\sqrt{t}}\Big|_{\pi/2}^{\pi} \right\rangle \\ &= \left\langle \frac{2}{3}, \frac{7\pi^3}{24} - \frac{3\pi^2}{8} + \left(3 - \frac{1}{\sqrt[3]{2}}\right)\sqrt[3]{\pi} + \ln 2, 2e^{\sqrt{\pi}} - 2e^{\sqrt{\pi/2}} \right\rangle .\end{aligned}$$

Indefinite integrals of vector valued functions are defined analogously to indefinite integrals of real-valued functions: If $\mathbf{r}(t)$ is a Riemann integrable function with $\mathbf{R}(t)$ an antiderivative defined over $\mathrm{Dom}(\mathbf{r})$, then, assuming $\mathrm{Dom}(\mathbf{r})$ is connected, a general antiderivative of $\mathbf{r}(t)$ is

$$\int \mathbf{r}(t)\,\mathrm{d}t = \mathbf{R}(t) + \mathbf{C}\,,$$

where $\mathbf{C}$ is a vector constant of integration. If the domain splits into several intervals, one may need separate constants of integration for each piece of the domain, as in the real-valued case.

Example. Find a general antiderivative of $\mathbf{r}(t) = \langle \sec^2 t, 1/1(t^2+1)\rangle$.

Solution:

$$\int \mathbf{r}(t)\,\mathrm{d}t = \langle \tan t + C_1, \arctan t + C_2\rangle\,,$$

provided t is confined to an interval of the form $(-\pi/2 + k\pi, \pi/2 + k\pi)$. If we assume $t \in \mathbb{R} - \{\pi/2 + k\pi \mid k \in \mathbb{Z}\}$, then we would need new constants across each interval of the form $(-\pi/2+k\pi, \pi/2+k\pi)$ to recover the most general possible antiderivative. Writing $\mathbf{R}(t) = \langle \tan t, \arctan t\rangle$, we could express this as

$$\int \mathbf{r}(t)\,\mathrm{d}t = \big\{\mathbf{R}(t) + \mathbf{C}_{k\pi} \text{ on } \ (-\pi/2 + k\pi, \pi/2 + k\pi)\big\}$$

where $\mathbf{C}_{k\pi}$ represents a constant dependent on the interval (and thus, on the choice of a shift factor $k\pi$ in the domain).

2.4. **Arc-length.** Regarding the tangent vector as a velocity vector for motion along a parameterized space curve, we may define *the speed* $\mathrm{d}s/\mathrm{d}t$ of a parameterized curve at time t to be the length of the velocity vector:

$$\frac{\mathrm{d}s}{\mathrm{d}t} = \|\dot{\mathbf{r}}(t)\| = \sqrt{\dot{\mathbf{r}}\cdot\dot{\mathbf{r}}}\,.$$

If we write $\mathbf{r}(t) = \langle f(t), g(t), h(t)\rangle$, then

$$\frac{\mathrm{d}s}{\mathrm{d}t} = \sqrt{\left(\frac{\mathrm{d}f}{\mathrm{d}t}\right)^2 + \left(\frac{\mathrm{d}g}{\mathrm{d}t}\right)^2 + \left(\frac{\mathrm{d}h}{\mathrm{d}t}\right)^2}\,.$$

Henceforth the notation will be streamlined (departing from the notations emphasized in the text). We may dispense with the labels f, g, and h for the component functions, and instead regard the coordinates x, y, and z as being given as functions of t along the curve described by $\mathbf{r}$. We also make use use Newton's fluxion notation, using an over-dot to represent time derivatives. We can then write the *arc-length differential* compactly in coordinates:

$$\mathrm{d}s = \sqrt{[\dot{x}(t)]^2 + [\dot{y}(t)]^2 + [\dot{z}(t)]^2}\,\mathrm{d}t$$

Integrating the arc-length differential over an interval $[a, b] \subseteq \mathrm{Dom}(\mathbf{r})$, one obtains the *arc-length* of the space curve $\mathbf{r}(t)$ as t ranges from a to b:

$$s(\mathbf{r}; [a, b]) = \int_a^b \sqrt{[\dot{x}(t)]^2 + [\dot{y}(t)]^2 + [\dot{z}(t)]^2}\,\mathrm{d}t$$

Example. Find the arc-length of the space curve $\mathbf{r}(t) = \langle e^t\cos 2t, e^t \sin 2t, 2e^t\rangle$ for $0 \leq t \leq \pi$.

Solution: First, we compute and simplify $\frac{\mathrm{d}s}{\mathrm{d}t}$:

$$\begin{aligned}\left(\frac{\mathrm{d}s}{\mathrm{d}t}\right)^2 &= (e^t\cos 2t - 2e^t\sin 2t)^2 + (e^t \sin 2t + 2e^t\cos 2t)^2 + 4e^{2t}\\ &= e^{2t}\cos^2 2t + 4e^{2t}\sin^2 2t - 4e^{2t}\cos 2t\sin 2t + e^{2t}\sin^2 2t + 4e^{2t}\cos^2 2t + 4e^{2t}\sin 2t\cos 2t + 4e^{2t}\\ &= e^{2t}(\cos^2 2t + 4\sin^2 2t + \sin^2 2t + 4\cos^2 2t + 4) = 9e^{2t}\,,\\ \frac{\mathrm{d}s}{\mathrm{d}t} &= 3e^t\,.\end{aligned}$$

Thus,

$$s\big(\mathbf{r}, [0, \pi]\big) = \int_0^\pi 3e^t\,\mathrm{d}t = 3e^\pi - 3\,.$$

We may assert that $0 \in \mathrm{Dom}(\mathbf{r})$ (one may always reparameterize so that this is true) and define the *arc-length function*:

$$s(t) := \int_0^t \mathrm{d}s(\tau) = \int_0^t \|\dot{\mathbf{r}}(\tau)\|\,\mathrm{d}\tau = \int_0^t \sqrt{[\dot{x}(\tau)]^2 + [\dot{y}(\tau)]^2 + [\dot{z}(\tau)]^2}\,\mathrm{d}\tau\,.$$

Often, there is no closed form for $s(t)$, but occasionally one is lucky enough to express s in terms of t in such a way that t can be solved for in terms of s. One can then reparameterize in terms of arc-length.

Example. Compute an arc-length parameterization of the helix $\mathbf{r}(t) = R(\cos(at)\hat{\mathbf{\imath}} + \sin(at)\hat{\mathbf{\jmath}}) + bt\hat{\mathbf{k}}$, where $R > 0$, a and b are all nonzero constants.

Solution: Observe that $\dot{\mathbf{r}}(t) = Ra(-\sin(at)\hat{\mathbf{\imath}} + \cos(at)\hat{\mathbf{\jmath}}) + b\hat{\mathbf{k}}$, and so $\frac{\mathrm{d}s}{\mathrm{d}t} = \|\dot{\mathbf{r}}(t)\| = \sqrt{R^2a^2 + b^2}$. Thus, $\mathrm{d}s = \sqrt{R^2a^2+b^2}\,\mathrm{d}t \implies s(t) = \sqrt{R^2a^2+b^2}\,t$. Solving for t, one may reparameterize:

$$\mathbf{r}(s) = R\cos\left(\frac{as}{\sqrt{R^2a^2+b^2}}\right)\hat{\mathbf{\imath}} + R\sin\left(\frac{as}{\sqrt{R^2a^2+b^2}}\right)\hat{\mathbf{\jmath}} + \frac{bs}{\sqrt{R^2a^2+b^2}}\hat{\mathbf{k}}\,.$$

In particular, note that when $1 = R = a = b$, we obtain the arc-length parameterization

$$\mathbf{r}(s) = \cos(s/\sqrt{2})\hat{\mathbf{\imath}} + \sin(s/\sqrt{2})\hat{\mathbf{\jmath}} + (s/\sqrt{2})\hat{\mathbf{k}}$$

of the unit counterclockwise circular helix encountered earlier.

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3. Motion in the Plane and Space

Throughout this section we will concern ourselves with motion: $\mathbf{r}(t)$ will denote a plane or space curve describing the position of an object at time t, in seconds. We'll presume units of meters for position, unless otherwise specified.

3.1. **Velocity and Acceleration.** We define the *instantaneous velocity* at time $t = t_0$ of an object undergoing motion along $\mathbf{r}(t)$ to be

$$\mathbf{v}(t_0) = \dot{\mathbf{r}}(t_0) = \frac{\mathrm{d}\mathbf{r}}{\mathrm{d}t}(t_0)\,.$$

The *acceleration* is then the next derivative,

$$\mathbf{a}(t_0) = \ddot{\mathbf{r}}(t_0) = \frac{\mathrm{d}\dot{\mathbf{r}}}{\mathrm{d}t}(t_0) = \frac{\mathrm{d}^2\mathbf{r}}{\mathrm{d}t^2}(t_0)\,.$$

Newton's famous second law linearly relates the net force $\mathbf{F}$ acting on the object to its acceleration with the mass m of the object as the constant of proportionality:

$$\mathbf{F} = m\mathbf{a}\,.$$

The study of kinematics is the study of motion, especially as can be derived from knowledge of forces acting on the object. We will soon use our tools of calculus with vector valued functions to study projectile motion and a general kinematic formula for motion in a uniform gravitational field.

Exercise 3.1. Find the velocity and acceleration of a particle whose position is $\mathbf{r}(t) = (t^2 - t)\hat{\imath} + (t^2 - t^3)\hat{\jmath} + (te^{-t} + e^t)\hat{\mathbf{k}}$.

Exercise 3.2. Find the velocity and acceleration of a particle which moves in the plane along the trajectory given by the cycloid $\mathbf{r}(t) = R\langle t - \sin t, 1 - \cos t\rangle$.

Exercise 3.3. Find the velocity and acceleration of a particle which moves with trajectory $\mathbf{r}(t) = 2\cot(t)\,\hat{\imath} + 2\sin^2(t)\,\hat{\jmath}$

3.2. **Position and The Net Change Principle.** Recall the net change principle, which is just a rephrasing of the fundamental theorem of calculus. We state it here for vector valued functions:

Proposition. *Given a rate of change* $\dot{\mathbf{r}}(t)$*, the net change of the function* $\mathbf{r}(t)$ *over the time period* $t_0 \le t \le t_1$ *is given by the definite integral of* $\dot{\mathbf{r}}(t)$ *over the interval* $[t_0, t_1]$*:*

$$\mathbf{r}(t_1) - \mathbf{r}(t_0) = \int_{t_0}^{t_1} \dot{\mathbf{r}}(t)\,\mathrm{d}t\,.$$

Now suppose we are given the velocity function $\mathbf{v}(t)$ of a particle, and know an initial position $\mathbf{r}_0 = \mathbf{r}(t_0)$. We can then recover the position as a function of time using the net change principle. Indeed, since velocity is the derivative of position, we deduce:

$$\mathbf{r}(t) = \mathbf{r}_0 + \int_{t_0}^{t} \mathbf{v}(\tau)\,\mathrm{d}\tau\,.$$

Exercise 3.4. Suppose a particle has velocity $\mathbf{v}(t) = \langle t\cos t + \sin t, \cos t - t\sin t, 3\cos^3 t - 9\sin^2 t\cos t\rangle$. Find the position function $\mathbf{r}(t)$ if at time $t = 0$, it has position $\mathbf{r}(0) = \langle 1, -1, 0\rangle$.

Exercise 3.5. Suppose the velocity $\mathbf{v}(t)$ of a particle traces out a circle on the unit sphere, and suppose $\frac{\mathrm{d}}{\mathrm{d}t}\|\mathbf{a}\| = 0$. What kinds of curves are the possible trajectories of the particle?

Exercise 3.6. A person is walking a dog on a 1.5m long leash. The person walks in a straight line at 1.4 meters per second. The dog, being both stubborn and curious about the smells along a hedge parallel to the path the person is walking, has to be dragged along, and maintains a position such that the leash is taut and the leash is in the tangential direction to the dogs motion. Parameterize the dog's walking path, and compute the velocity and acceleration as functions of time. The curve of the dog's path is called a *tractrix* or a *hundkurve*.

3.3. **Projectile Motion.** For this section we will consider the motion of projectiles which are launched with an initial velocity $\mathbf{v}_0 = \mathbf{v}(0)$, after which the only force acting on the projectile is assumed to be that of a uniform gravitational field. We neglect considerations such as wind and air resistance for simplicity (these complicate the differential equations of motion sufficiently that we'd be reliant on numerical analysis, as the extent of the effects of wind and air resistance are dependent on the particular shape of the projectile).

By uniform gravity, we assume that the force of gravity is constant on a given object, according to Newton's second law:

$$\mathbf{F} = m\,\mathbf{a} = -mg\,\hat{\mathbf{k}}\,,$$

where $g = 9.80665\,\mathrm{m/s^2}$ is the average magnitude of acceleration due to gravity on the surface Earth. The actual value of the magnitude of acceleration varies depending on location, in part because earth's mass is not uniformly distributed. We'll generally use the approximate value $g = 9.8\,\mathrm{m/s^2}$.

Assume the projectile has an initial position sitting at some height h above the origin, and we are given an initial velocity $\mathbf{v}_0$ which makes some angle of inclination with the horizontal and has magnitude $v_0 = \|\mathbf{v}_0\|$. Since gravitation is assumed constant, integrating the constant acceleration will give us a a linear velocity:

$$\mathbf{v}(t) - \mathbf{v}_0 = \int_0^t \mathbf{a}\,\mathrm{d}\tau = \mathbf{a}t \implies \mathbf{v}(t) = \mathbf{a}t + \mathbf{v}_0\,.$$

Let θ be the angle between the projection of $\mathbf{v}_0$ into the xy-plane, and the x-axis, and let φ be the angle of incline, made between $\mathbf{v}_0$ and it's xy-plane projection. Thus, we may express the initial velocity as

$$\mathbf{v}_0 = v_0\Big(\cos(\theta)\cos(\varphi)\,\hat{\imath} + \sin(\theta)\cos(\varphi)\,\hat{\jmath} + \sin(\varphi)\,\hat{\mathbf{k}}\Big)\,.$$

One way to simplify things is to rotate our coordinates so that the x axis is along the direction of horizontal motion, and thus, $\theta = 0$. Then, we can write everything with just $\hat{\imath}$ and $\hat{\mathbf{k}}$ components (or $\hat{\imath}$ and $\hat{\jmath}$ components, if you rotate your coordinate system again, so that $\hat{\jmath}$ is the vertical direction from the ground), or use angle brackets with just two entries.

With such a simplification, the velocity function of the projectile becomes

$$\mathbf{a}t + \mathbf{v}_0 = v_0\cos(\varphi)\,\hat{\imath} + \big(v_0\sin(\varphi) - gt\big)\hat{\mathbf{k}}\,.$$

We can integrate again, using $\mathbf{r}_0 = h\hat{\mathbf{k}}$ to obtain the position:

$$\begin{aligned}\mathbf{r}(t) - \mathbf{r}_0 &= \int_0^t \mathbf{v}(\tau)\,\mathrm{d}\tau = \int_0^t \mathbf{a}\tau + \mathbf{v}_0\,\mathrm{d}\tau\\ &= \frac{1}{2}\mathbf{a}t^2 + \mathbf{v}_0 t\\ \implies \mathbf{r}(t) &= \frac{1}{2}\mathbf{a}t^2 + \mathbf{v}_0 t + \mathbf{r}_0\\ &= v_0\cos(\varphi)t\,\hat{\imath} + \big(h + v_0\sin(\varphi)t - gt^2/2\big)\hat{\mathbf{k}}\,.\end{aligned}$$

Such a formula for $\mathbf{r}(t)$ is called a *general kinematic equation*. Using units of meters and seconds, it can be expressed in angle brackets as

$$\mathbf{r}(t) = \langle v_0\cos(\varphi)t, -4.9t^2 + v_0\sin(\varphi)t + h\rangle\,.$$

If instead we switch from meters to feet, g becomes approximately $32\mathrm{ft/s^2}$, and the equation is instead

$$\mathbf{r}(t) = \langle v_0\cos(\varphi)t, -16t^2 + v_0\sin(\varphi)t + h\rangle\,.$$

From our general kinematic equation we observe that the motion decomposes into linear horizontal motion, and quadratic vertical motion. The resulting trajectory curve is a parameterized parabola. We can easily extract information such as the maximum height, duration of flight, and velocity on impact using tools of algebra and/or elementary calculus.

Example. A projectile is launched at 16m/s at an angle of 60.0° from the horizontal and a height of 2.0 meters. Find the maximum height, the horizontal range , and total flight duration of the projectile.

Solution: From the above considerations, we have

$$\mathbf{r}(t) = \langle 8t, -4.9t^2 + 8\sqrt{3}t + 2.0\rangle = 8t\hat{\imath} + (-4.9t^2 + 8\sqrt{3} + 2)\,\hat{\mathbf{k}}\,.$$

The maximum height occurs at the vertex of the parabola, which may be found algebraically if one has memorized such a description. One can also find maximum height by determining when the vertical velocity vanishes, and evaluating the vertical component at that time. Then

$$\mathbf{v}(t)\cdot\hat{\mathbf{k}} = -9.8t + 8\sqrt{3} = 0 \implies t = 8\sqrt{3}/9.8 \approx 1.41\,,$$

so the maximum height is achieved about 1.41 seconds into the flight. This gives a maximum height of

$$\mathbf{r}(1.41)\cdot\hat{\mathbf{k}} \approx 11.8\,\mathrm{m}\,.$$

To find the flight duration, we seek positive t such that $\mathbf{r}(t)\cdot\hat{\mathbf{k}} = 0$. Applying the quadratic formula gives $t \approx 2.97$ seconds. The horizontal range is then found by computing $\mathbf{r}(2.97)\cdot\hat{\imath} = 8(2.97) \approx 23.7$, so the horizontal range of the projectile is approximately 23.7 m.

Exercise 3.7. Find the velocity and position functions of a projectile with initial velocity $\mathbf{v}_0 = (5.0\hat{\imath} + 10.0\hat{\jmath} + 15.0\hat{\mathbf{k}})\,\mathrm{m/s}$ which is launched 4.0 meters above the ground. What is its maximum height? How long does it take to hit the ground? What is its final velocity $\mathbf{v}_f$ as it impacts the ground?

Exercise 3.8. Prove that $\varphi = \pi/4$ maximizes the horizontal range of a projectile, and express this range in terms of the initial speed v_0.

Exercise 3.9. Let α, β, and γ be the angles between $\mathbf{v}_0$ and $\hat{\imath}$, $\hat{\jmath}$ and $\hat{\mathbf{k}}$, respectively. Compute a general position function like the one above in terms of the initial velocity expressed using α, β, γ, and initial speed v_0 for a projectile launched from $\mathbf{r}_0 = h\hat{\mathbf{k}}$.

Exercise 3.10. Suppose $\mathbf{r}(t)$ describes the motion of a particle in $\mathbb{R}^3$ with a constant acceleration $\mathbf{a}$, initial velocity $\mathbf{v}_0$, and initial position $\mathbf{r}_0$. Show that $\mathbf{r}(t)$ is a parameterized parabola, and express the the arc-length function $s(t)$ measuring distance traveled along the trajectory from $\mathbf{r}_0$ to $\mathbf{r}(t)$ as an integral in terms of $\mathbf{a}$ and $\mathbf{v}_0$.

For a discussion of tangential and normal components of acceleration, and to learn about curvature and torsion, and explore the vector geometry of planetary orbits, see the next batch of notes: *Curvature, Natural Frames, and Acceleration for Plane and Space Curves.*

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4. Hints and Answers to Some Exercises

4.1. Describing Curves Parametrically.

1.1 Consider both direction and duration. If you draw a unit circle and label a point, you should be able to recognize the values of the parameters of the different parameterizations which give this point as arising from considering different angles.

1.2 Consider the relationship between the distance rolled by the disk, and the angle made between $-\hat{\jmath}$ and the displacement vector from the wheel's center to the point $\mathbf{r}(t)$. Draw an appropriate right triangle to obtain the x and y components of the parameterization from a careful diagram of the disk mid-roll. The horizontal speed of the disk's center is R meters per second.

1.3 To get the back and forth motion, try using a trigonometric function in place of t. You need to calibrate the period based on the length of the line segment.

4.2. Vector Valued Functions.

1.4 Consider how one decides if a given $t \in \mathbb{R}$ is in the natural domain of a component–under what conditions on components is $\mathbf{r}$ defined at t?

1.5 For the hyperbola, try to see geometrically that hyperbolic trigonometric functions play a natural role analogous to trigonometric functions. It is helpful to remember the hyperbolic analogue of of the pythagorean identity: what relation do $\sinh^2 t$ and $\cosh^2 t$ share?

1.6 For b, you can reparameterize to understand how the x and y components are related by a polynomial equation. For the witch of Agnesi, Pythagorus is your friend.

1.7 To show that they are ellipses, try writing Cartesian equations and either eliminating a variable (such as z) or defining a new variable that relates x and z. You can also use the classical definition of an ellipse as the loci of all points P such that the sums of the distances $|PF_1|$ and $|PF_2|$ equals some constant $2a$, for a pair of fixed points F_1 and F_2, called foci.

1.8 First figure out the radius R of the circle. Can you find a pair of perpendicular vectors of length R, lying in the desired plane? (Extra hint: Use a cross product!)

4.3. Limits.

2.1 Remember that the limits at infinity of an algebraic function are determined by the highest degree terms of the numerator and denominator. Be careful of the sign! You should get that the limit equals $\frac{1}{2}(\pi\hat{\mathbf{j}} - 3\hat{\mathbf{k}})$.

2.2 Use limits to help sketch the curves. It also helps to algebraically simplify, and sometimes to reparameterize.

4.4. Differentiation.

2.3 You can work in components, and appeal to the familiar properties of derivatives of real-valued functions.

4.5. Integration.

2.6

Definition. Let $\mathscr{P}$ denote a partition $a = t_0 < t_1 < \ldots < t_n = b$ of $[a, b]$, with a choice of sample points $t_i^* \in [t_{i-1}, t_i]$ for each subinterval, $i = 1, \ldots, n$. The partition is uniform if the subinterval lengths $\Delta t_i := t_i - t_{i-1}$ are all equal, in which case we can denote this length by Δt. The norm $\|\mathscr{P}\|$ of the partition is the maximum subinterval length: $\|\mathscr{P}\| := \max\{\Delta t_i \,|\, i = 1 \ldots n\}$. Observe that $\Delta t = (b - a)/n = \|\mathscr{P}\|$ in the uniform case. We define a Riemann sum of $\mathbf{r}(t)$ on $[a, b]$ to be

$$S(\mathbf{r}, \mathscr{P}) = \sum_{i=1}^{n} \mathbf{r}(t_i^*)\Delta t_i \,.$$

Choosing a sequence $\mathscr{P}_k$ of partitions with norms decreasing to 0 and sample points, one defines the Riemann integral

$$\int_a^b \mathbf{r}(t)\,\mathrm{d}t = \lim_{k\to\infty} S(\mathbf{r}, \mathscr{P}_k)\,.$$

One interpretation of the terms of a vector valued Riemann sum is as a vector of rectangular areas. A more physical-geometric interpretation is as follows: if $\mathbf{r}(t)$ is a velocity curve, then $\mathbf{r}(t_i^*)\Delta t_i$ approximates the net displacement of the particle during the interval $[t_{i-1}, t_i]$. Indeed, by the mean value theorem, the average velocity is attained at some point $\bar{t}_i$ in that interval, and as $\|\mathscr{P}_k\| \to 0$, the sample points become closer to $\bar{t}_i$, and the quantity $\mathbf{r}(t_i^*)$ approaches the average velocity, meaning $\mathbf{r}(t_i^*)\Delta t_i \approx \mathbf{r}(\bar{t}_i)\Delta t_i$ is the approximate net displacement.

4.6. **Position and the Net Change Principle.**

3.5 Observe that $\mathbf{v} \perp \mathbf{a}$, and since $\|\mathbf{a}\|$ is constant, so is $\|\mathbf{v}\|$. It is also helpful to consider the normal to the plane cutting out the circle, and the plane's distance from the origin. You should be able to show that the position function describes either a helix or a circle.