The Banana Tension Problem:

You have a static banana, with 120 grams mass, suspended from the ceiling by two taught wires:

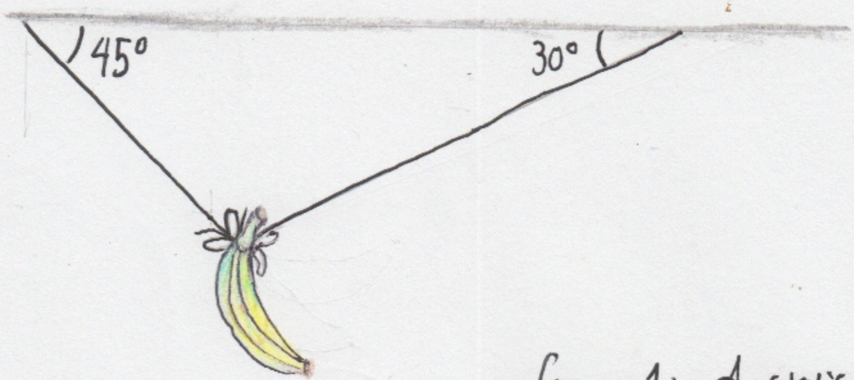


figure 1: A static, suspended banana.

One wire makes a 45° angle with the ceiling, the other, 30°. The task is to determine the tension in the wires.

Tension is a <u>force</u>. Newton tells us that force, as a vector quantity, is proportional to acceleration, via mass:

(1) $$\vec{F} = m\vec{a}.$$

If the banana were in freefall, the force, ignoring air-resistance (which depends on velocity) is gravitational force:

(2) $$\vec{F}_g = m\vec{g} = \left(-9.8\, m/s^2 \cdot m\right)\hat{j},$$

where $\hat{j}$ is the vertical unit vector in coordinates centered on the banana's center of mass. See below:

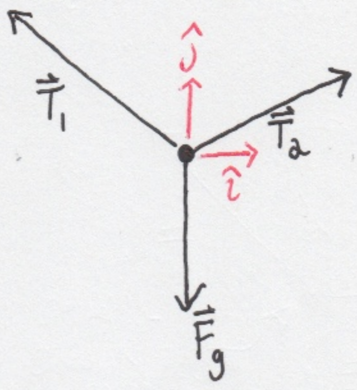
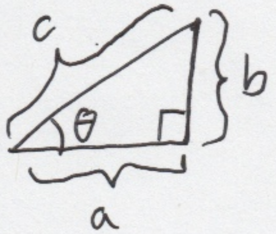


Figure 2: A Free body force diagram for the banana's center of mass, with vectors $\hat{i}, \hat{j}$ labeled to provide a useful rectangular coordinate system. Note $\vec{F}_g$ is parallel but in the opposite direction to $\hat{j}$.

Since we are concerned with tension, a force, we will use the SI. unit of force, the Newton. 1 Newton, abbr. 1N, is the force required to accelerate a 1 kilogram mass 1 meter per squared seconds: $1N = \frac{1kg \cdot 1m}{s^2}$. Thus, using that gravity accelerates objects downward at $9.8\,m/s^2$ near earth's surface,

$$(3) \quad \vec{F}_g = -(9.8\,m/s^2 \cdot 0.120\,kg) = -(1.176\,N)\hat{j}.$$

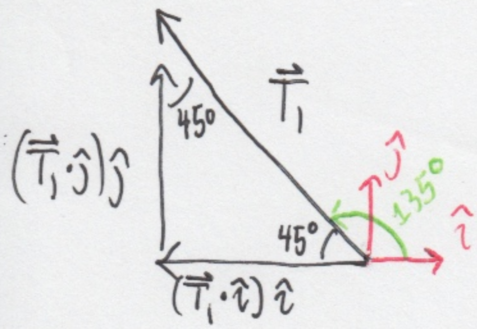
It will be convenient to decompose the tensions $\vec{T}_1$ and $\vec{T}_2$ into sums of vectors in the $\pm\hat{i}$ and $\pm\hat{j}$ directions, using trigonometry. Recall, given a right triangle:



$$a^2 + b^2 = c^2 \qquad \sin\theta = \frac{b}{c} \qquad \cos\theta = \frac{a}{c}$$

$$\tan\theta = \frac{b}{a}$$

This will be used repeatedly.



There are two ways to arrive at the decomposition of $\vec{T}_1$ into vertical and horizontal components. First, using the preceding trigonometric identities, we note that the horizontal component should have magnitude $\|\vec{T}_1\| \cos 45°$, since $\|\vec{T}_1\|$ is the magnitude of the hypotenuse vector. Since the horizontal component is parallel but opposite to $\hat{\imath}$, we deduce it is given by $-\|\vec{T}_1\| \cos 45° \, \hat{\imath}$. Similarly, the vertical component can be seen to be $\|\vec{T}_1\| \sin 45° \hat{\jmath}$. Thus

$$(4) \quad \vec{T}_1 = -\|\vec{T}_1\| \cos 45° \hat{\imath} + \|\vec{T}_1\| \sin 45° \hat{\jmath}$$

$$= \frac{\|\vec{T}_1\|\sqrt{2}}{2} \left(-\hat{\imath} + \hat{\jmath} \right).$$

The other way to arrive at this decomposition is to employ dot products. Since, for any two vectors, $\vec{a}, \vec{b}$, the dot product satisfies the geometric identity $\vec{a} \cdot \vec{b} = \|\vec{a}\| \|\vec{b}\| \cos \theta$, where θ is the angle between $\vec{a}$ and $\vec{b}$, one has

$$(5) \quad \vec{T}_1 = (\vec{T}_1 \cdot \hat{\imath})\hat{\imath} + (\vec{T}_1 \cdot \hat{\jmath})\hat{\jmath} = \|\vec{T}_1\| \|\hat{\imath}\| \cos 135° \hat{\imath} + \|\vec{T}_1\| \|\hat{\jmath}\| \cos 45° \hat{\jmath}$$

$$= \|\vec{T}_1\| \left(-\frac{\sqrt{2}}{2}\right) \hat{\imath} + \|\vec{T}_1\| \left(\frac{\sqrt{2}}{2}\right) \hat{\jmath}.$$

Similarly, we can decompose $\vec{T}_2$ into vertical and horizontal components:

$$(6)\quad \vec{T}_2 = \|\vec{T}_2\| \cos(30^\circ)\hat{\imath} + \|\vec{T}_2\| \sin(30^\circ)\hat{\jmath}$$

$$= \|\vec{T}_2\| \left(\frac{\sqrt{3}}{2}\hat{\imath} + \frac{1}{2}\hat{\jmath} \right).$$

Now, the key to solving for the tensions lies in finding the magnitudes $\|\vec{T}_1\|$ and $\|\vec{T}_2\|$ by applying Newton's force law. Since the system is static, there is no <u>net</u> acceleration. Thus the sum of these vectors is $\vec{0}$

$$(7)\quad \vec{F}_g + \vec{T}_1 + \vec{T}_2 = \vec{0}.$$

Using components:

$$(8)\quad -1.176\,N\hat{\jmath} - \|\vec{T}_1\|\left(\frac{\sqrt{2}}{2}\right)\hat{\imath} + \|\vec{T}_1\|\left(\frac{\sqrt{2}}{2}\right)\hat{\jmath} + \|\vec{T}_2\|\left(\frac{\sqrt{3}}{2}\right)\hat{\imath} + \|\vec{T}_2\|\left(\frac{1}{2}\right)\hat{\jmath} = \vec{0}$$

The $\hat{\imath}$ components yield the equation

$$(9)\quad -\|\vec{T}_1\| \frac{\sqrt{2}}{2} + \|\vec{T}_2\| \frac{\sqrt{3}}{2} = 0 \iff \|\vec{T}_1\| = \sqrt{\frac{3}{2}}\,\|\vec{T}_2\|$$

$$= \frac{\sqrt{6}}{2}\|\vec{T}_2\|$$

The equation for the $\hat{j}$ components yields

$$(10) \quad -1.176N + \underbrace{\frac{\sqrt{2}}{2}\|\vec{T}_1\|}_{\frac{\cancel{\sqrt{2}}}{2}\frac{\sqrt{3}}{\cancel{\sqrt{2}}}\|\vec{T}_2\|} + \frac{1}{2}\|\vec{T}_2\| = 0$$

(substituting).

$$(11) \quad \left(\frac{\sqrt{3}}{2} + \frac{1}{2}\right)\|\vec{T}_2\| = 1.176N$$

$$\|\vec{T}_2\| = 0.86N$$

(divide, calculate, and take 2 significant digits, since our value of $g = 9.8\,m/s^2$ gives only 2.)

Thus, $\|\vec{T}_1\| = 1.05N$, and

$$\boxed{\begin{aligned} (12) \quad \vec{T}_1 &= -\frac{1.05N\,\hat{i} + 1.05N\,\hat{j}}{\sqrt{2}} = -0.75N\,\hat{i} + 0.75N\,\hat{j} \\ T_2 &= \frac{0.86N\sqrt{3}}{2}\hat{i} + \frac{0.86N}{2}\hat{j} = 0.75N\,\hat{i} + 0.43N\,\hat{j}. \end{aligned}}$$