Notes on Double Integrals

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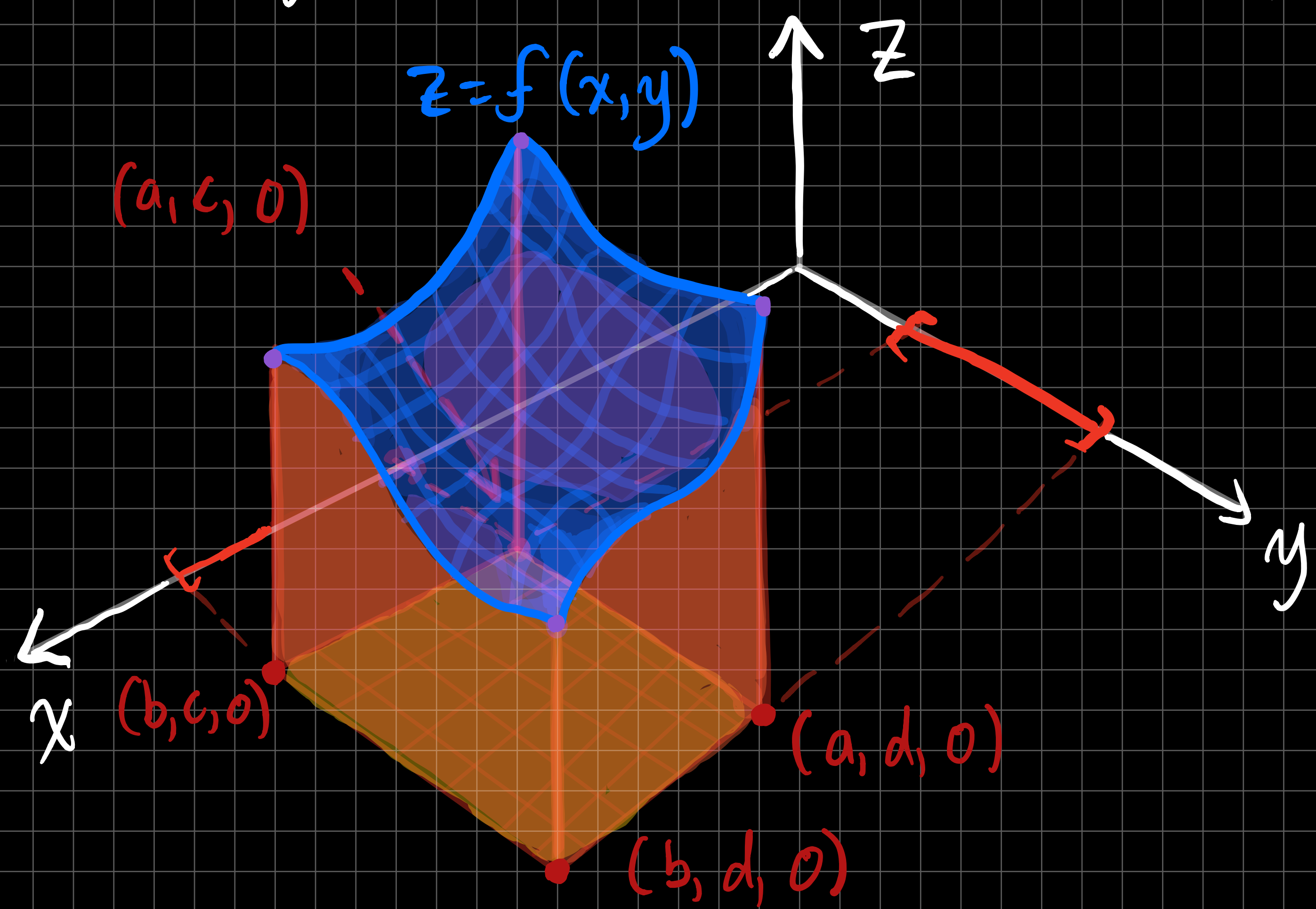
Temporarily assume that $f(x,y)$ is both continuous & positive over some rectangular region

$$R = [a,b] \times [c,d] = \{(x,y) : a \le x \le b,\ c \le y \le d\}$$

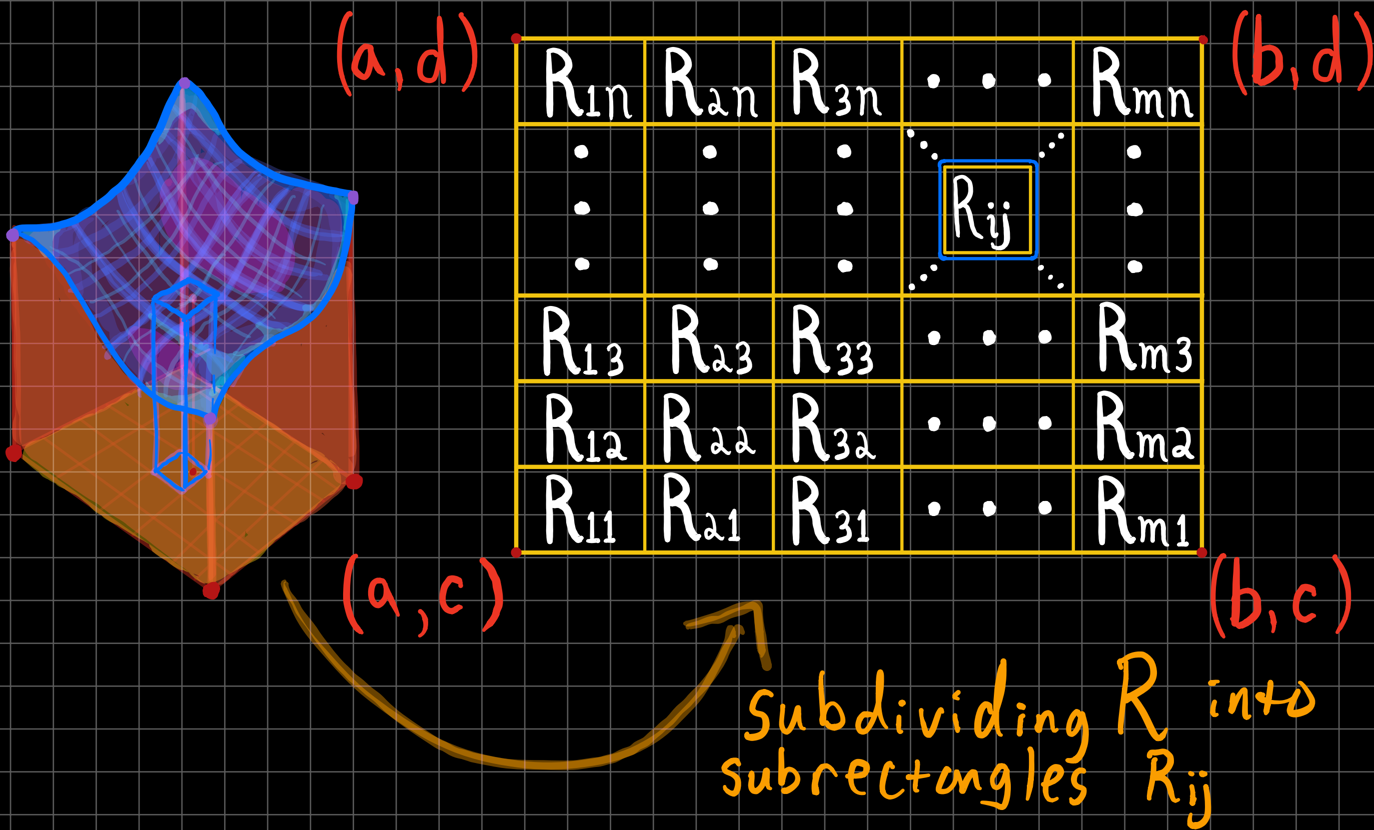
The graph $z = f(x,y)$ can be imagined as the "roof" over a room with floor R (placed in the xy plane of $\mathbb{R}^3$), and walls which are regions in the planes $x=a$, $x=b$, $y=c$, & $y=d$:



We want to define the double integral of $f(x,y)$ over R, denoted by $\iint\limits_R f(x,y)\,dA$ to be the Volume of this room.

To arrive at a careful definition, we will employ Riemann Sums.

The idea is to approximate our room using boxes, whose volumes are readily calculated. Then we increase the accuracy of our approximation by using more boxes. In the limit, we obtain the Riemann Sum definition of the double integral.



Setup for Riemann Sums:
(We'll use <u>uniform partitions</u> for each interval, $[a,b]$ & $[c,d]$)

- Fix positive integers m, n.
- Set $\Delta x = \frac{b-a}{m}$, $x_0 = a$,

  $x_i = x_0 + \Delta x = a + i\Delta x$,

  $\Delta y = \frac{d-c}{n}$, $y_0 = c$,

  $y_j = y_0 + \Delta y = c + j\Delta y$.

Then the ij^{th} subrectangle is

$$R_{ij} = [x_{i-1}, x_i] \times [y_{j-1}, y_j]$$

= Subjectangle in ith column to the right of $x=a$ & jth row above $y=c$.

- Pick a point $(x_{ij}^*, y_{ij}^*) \in R_{ij}$ for each R_{ij}

The point (x_{ij}^*, y_{ij}^*) is called a <u>sample point</u>, and determines the height of the ij^{th} box, via evaluation of $f(x,y)$.

The volume of the ij^{th} box is

$$V_{ij} = \underbrace{f(x_{ij}^*, y_{ij}^*)}_{\text{box height}} \underbrace{\Delta x \Delta y}_{= \Delta A}$$

Thus $V \approx \sum_{i=1}^{m} \sum_{j=1}^{n} f(x_{ij}^*, y_{ij}^*) \Delta A$

The double integral will be defined as a limit of this. We will no longer assume $f(x,y) > 0$ or continuous on R.

Definition: Let $f(x,y)$ be a function defined over the rectangle $R = [a,b] \times [c,d]$. Then the double integral of $f(x,y)$ over R is

$$\iint_R f(x,y)\,dA = \lim_{\substack{m\to\infty \\ n\to\infty}} V[f,R,m,n]$$

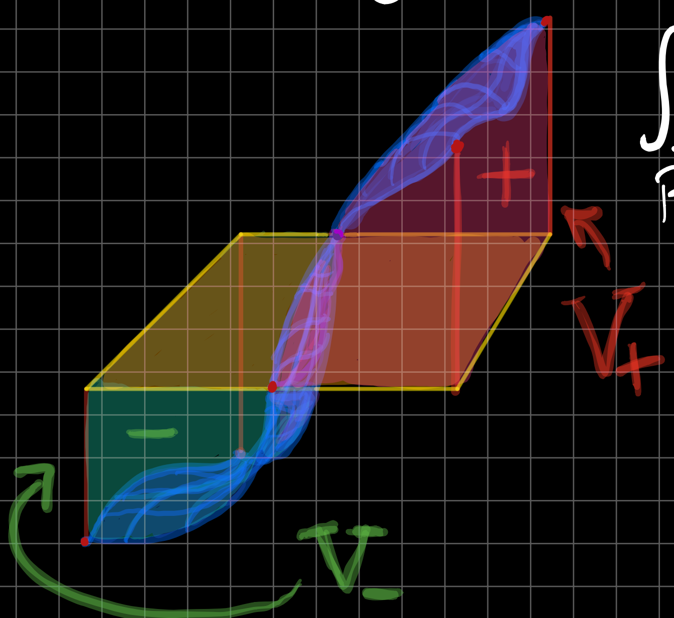
where $V[f,R,m,n] = \sum_{i=1}^{m} \sum_{j=1}^{n} f(x_{ij}^*, y_{ij}^*)\Delta A$, provided the limit exists.

Note: for $f(x,y)$ continuous over R, the limit always exists. Whenever $\iint_R f(x,y)\,dA$ exists, we say f is integrable over R.

Remarks: 1.) We did not assume $f(x,y) > 0$ for the definition. If f is positive & continuous we can interpret $\iint_R f(x,y)\,dA$ as the geometric volume bounded below $z = f(x,y)$, above the plane $z=0$, and within the walls $x=a$, $x=b$, $y=c$, $y=d$.

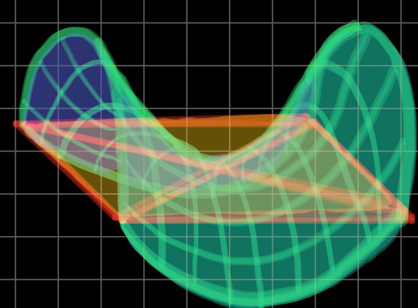
On the other hand, if f takes non positive values in R, we may interpret the double integral as "Signed Volume/net Volume":

$$\iint_R f(x,y)\,dA = V_+ - V_-$$



e.g. $\iint_{[-1,1]\times[-1,1]} x^2 - y^2 \, dA = 0,$

As above $z=0$, so below!



2.) If $\iint_R f(x,y)\,dA$ exists, then its value is independent of the method of choosing sample points.

In particular, we may choose sample points as corners or even midpoints of rectangles. Midpoints yield a decent numerical approximation of $\iint_R f(x,y)\,dA$:

<u>The Midpoint rule:</u>

$$\iint_R f(x,y)\,dA \approx \sum_{i=1}^{m}\sum_{j=1}^{n} f(\bar{x}_i, \bar{y}_j)\,\Delta x\,\Delta y$$

where $(\bar{x}_i, \bar{y}_j) = \left(\frac{x_{i-1}+x_i}{2}, \frac{y_{j-1}+y_j}{2}\right)$

is the midpoint of R_{ij}.

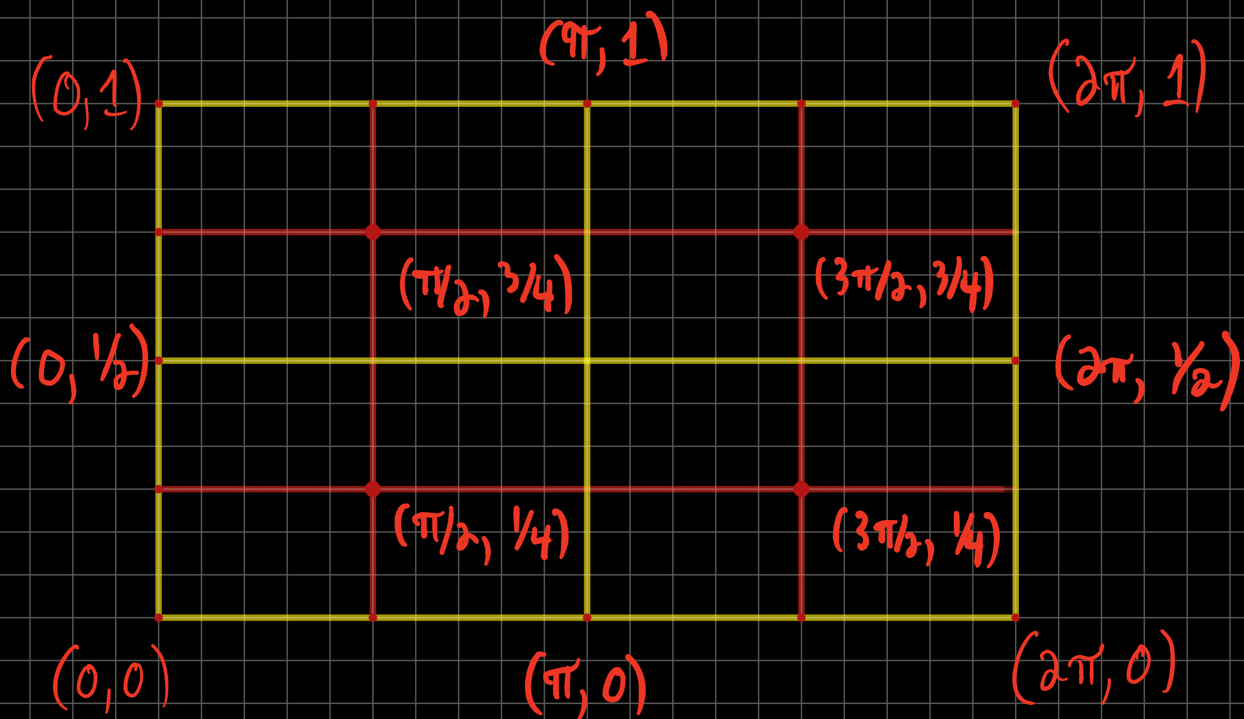
Exercise: Generalize the trapezoid rule to a "tangent plane approximation rule" for numerically approximating $\iint_R f(x,y)\,dA$.

3.) If $f(x,y) = 1$ throughout R, we recover the area of the rectangle R:

$$\iint_R 1\,dA = A(R) = (b-a)\cdot(d-c).$$

Ex: Use a Midpoint sum to approximate

$$\iint_{[0,2\pi]\times[0,1]} \sin xy \, dA \text{ using } 4 \text{ rectangles.}$$



$$\iint_{[0,2\pi]\times[0,1]} \sin xy \, dA \approx \underbrace{\frac{\pi}{2}}_{\Delta A} \cdot \left(\sin\frac{\pi}{8} + 2\sin\frac{3\pi}{8} + \sin\frac{9\pi}{8} \right)$$

By the sine half angle identity:

$$\sin\left(\frac{\pi}{8}\right) = \sqrt{\frac{1-\cos(\pi/4)}{2}} = \frac{\sqrt{2 \quad \sqrt{2}}}{2}$$

$$= -\sin\left(\frac{9\pi}{8}\right)$$

$$\sin\left(\frac{3\pi}{8}\right) = \frac{\sqrt{2+\sqrt{2}}}{2}.$$

Thus: $\displaystyle\iint_{[0,2\pi]\times[0,1]} \sin xy \, dA \approx \frac{\sqrt{2}+\sqrt{2}}{2}\pi.$

Question: How do we compute $\iint_R f(x,y)\,dA$ exactly?

Answer: Iterated integrals. To develop this answer, we first introduce partial integration.

Definition: The indefinite partial integral of $f(x,y)$ with respect to x is the class of general x-antiderivatives, represented as

$$\int f(x,y)\,dx = F(x,y) + G(y)$$

where $\frac{\partial G}{\partial x} \equiv 0$ & $\frac{\partial F}{\partial x} = f(x,y)$.

By the Fundamental Theorem of Calculus, we may take $F(x,y)$ to be an antiderivative of $f(x,y)$ obtained by definite integration:

$$F(x,y) = \int_a^x f(t,y)\, dt, \text{ y held constant.}$$

Indeed: $\frac{\partial F}{\partial x} = \frac{\partial}{\partial x} \int_a^x f(t,y)\,dt = f(x,y)$

by F.T.C. I.

We also have definite partial integrals which satisfy F.T.C. II:

$$\int_a^b f(x,y)\, dx = F(b,y) - F(a,y).$$

Similarly: $\int g(x,y)\, dy = G(x,y) + D(x)$,

$\frac{\partial D}{\partial y} \equiv 0$, $\frac{\partial G}{\partial y} = g(x,y)$, and

$$\int_c^d g(x,y)\, dy = G(x,d) - G(x,c).$$

<u>Example:</u> $\int_0^2 x+y\; dy = \left[xy + \frac{1}{2}y^2\right]_0^2$

$$= 2x + 2.$$

Thus observe that a definite partial integral may in general be a function of the remaining (un-integrated) variables.

<u>Example</u>: $\int_0^\pi y\sin(xy)\,dx = \left[-\cos(xy)\right]_0^\pi$

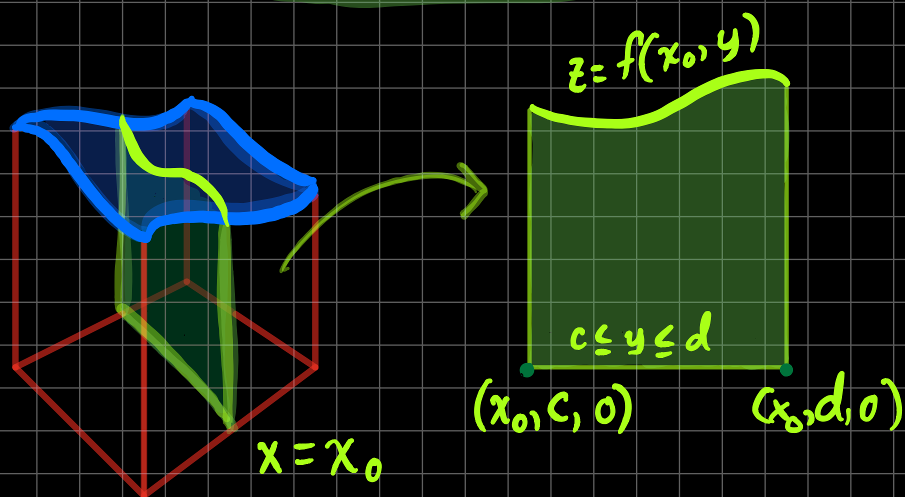
$$= -\cos(\pi y) + 1.$$

We again temporarily assume $f(x,y) > 0$ and continuous over the rectangle $R = [a,b] \times [c,d]$.

Let S be the solid bounded by

$z = f(x,y)$, $z = 0$, $x = a$, $x = b$, $y = c$, & $y = d$.

Slice S by the plane $x = x_0 \in [a,b]$; observe that the (net) area of the slice is $A(x_0) = \int_c^d f(x_0, y)\, dy$.



Thus, definite partial integrals have the geometric interpretation of computing slice areas as a function of the slice plane coordinate.

By the cross sectional area principle:

$$V = \mathrm{Vol}(S) = \int_a^b A(x)\, dx$$

$$(*) \qquad = \int_a^b \left[\int_c^d f(x,y)\,dy \right] dx$$

provided that $\iint_R f(x,y)\,dA$ and the above <u>iterated integral</u> $(*)$ both exist (and they will, since we assumed continuity).

There's no reason to prefer to slice S along planes of constant x; we may also slice along planes of constant y to obtain

$$V = \int_c^d \left[\int_a^b f(x,y)\,dx \right] dy.$$

Fubini's Theorem: If f is integrable over a rectangle R, and both iterated integrals exist, then

$$\iint_R f(x,y)\,dA = \int_a^b \int_c^d f(x,y)\,dy\,dx$$

$$= \int_c^d \int_a^b f(x,y)\,dx\,dy.$$

In particular, if f is continuous, or bounded with "nice enough" discontinuities (e.g. jump discontinuities along only finitely many curves), then the above equations hold.

Hard Exercise: Let $R = [0,1] \times [0,1]$, and define

$$f(x,y) = \begin{cases} 1 & \text{if } x \text{ is rational} \\ 2y & \text{if } x \text{ is irrational} \end{cases}$$

$$g(x,y) = \begin{cases} \dfrac{x^2-y^2}{(x^2+y^2)^2} & \text{if } (x,y)\neq(0,0) \\ 0 & \text{if } (x,y)=(0,0). \end{cases}$$

Explain in each case why Fubini's Theorem does not apply.

Determine precisely how the Theorem's conclusions fail in each case by computing iterated integrals where possible.

Example: $\iint_R y\sin(xy)\,dA$, $R=[1,2]\times[0,\pi]$

We have 2 possible orders to set up iterated integrals, but 1 order requires integration by parts!

$$\iint_R y\sin(xy)\,dA = \int_1^2\int_0^\pi y\sin(xy)\,dy\,dx$$
$$= \int_0^\pi\int_1^2 y\sin xy\,dx\,dy.$$

The second order is easier, requiring only a simple substitution: $u = xy$, $du = y\,dx$.

$$\iint_R y\sin(xy)\,dA = \int_0^{\pi}\int_1^2 y\sin(xy)\,dx\,dy$$

$$= \int_0^{\pi}\left[-\cos(xy)\Big|_1^2\right]dy$$

$$= \int_0^{\pi}\cos y - \cos 2y\,dy$$

$$= \left[\sin y - \frac{1}{2}\sin 2y\right]_0^{\pi} = 0.$$

Average Value

Definition: The average value of a function $f(x,y)$ over a rectangle R is

$$\bar{f}(R) := \frac{1}{A(R)}\iint_R f(x,y)\,dA$$

$$= \frac{1}{(b-a)(d-c)}\iint_{[a,b]\times[c,d]} f(x,y)\,dA.$$

Example: Find $\bar{f}(R)$ for $f(x,y) = xy^2$,
$R = [0,2] \times [0,3]$.

$$\bar{f}(R) = \frac{1}{2 \cdot 3} \iint_R xy^2 \, dA$$

$$= \frac{1}{6} \int_0^2 \int_0^3 xy^2 \, dy \, dx$$

$$= \frac{1}{6} \int_0^2 \frac{1}{3} xy^3 \Big|_0^3 \, dx$$

$$= \frac{1}{6} \int_0^2 9x \, dx = \frac{1}{6} \left[\frac{9}{2} x^2 \right]_0^2$$

$$= 3.$$

A Trick: If $f(x,y) = u(x)v(y)$, then

$$\iint_R f(x,y) \, dA = \left(\int_a^b u(x) \, dx \right) \left(\int_c^d v(y) \, dy \right).$$

e.g. $6\bar{f}(R) = \left(\int_0^2 x \, dx \right) \left(\int_0^3 y^2 \, dy \right) = 2 \cdot 9$

$\bar{f}(R) = \frac{18}{6} = 3.$

Basic Properties of Double integrals:

1.) Linearity: If f, g are each integrable over R, and $c \in \mathbb{R}$ is any constant

$$\iint_R c f(x,y) + g(x,y)\, dA = c\iint_R f(x,y)\, dA + \iint_R g(x,y)\, dA.$$

2.) If $f(x,y) \geq g(x,y)$ throughout R:

$$\iint_R f(x,y)\, dA \geq \iint_R g(x,y)\, dA.$$

3.) In particular, suppose $f(x,y)$ is continuous over R, and let $m = \min_{(x,y)\in R} f(x,y)$, $M = \max_{(x,y)\in R} f(x,y)$. Then since

$m \leq f(x,y) \leq M$, $m \cdot A(R) \leq \iint_R f(x,y)\, dA \leq M \cdot A(R)$,

where $A(R) = \text{area}(R)$.

eg. one can show $0 < \iint_{[0,1]\times[0,1]} \ln(2xy - x - y + 2)\, dA \leq \ln 2$

by showing that $\max_R f(x,y) = \ln 2$ (@ $(0,0)$ & $(1,1)$)

and $\min_R f(x,y) = 0$ (@ $(1,0)$ & $(0,1)$)

Using the inequality above, together with the method given previously for finding extrema of continuous functions in compact domains.