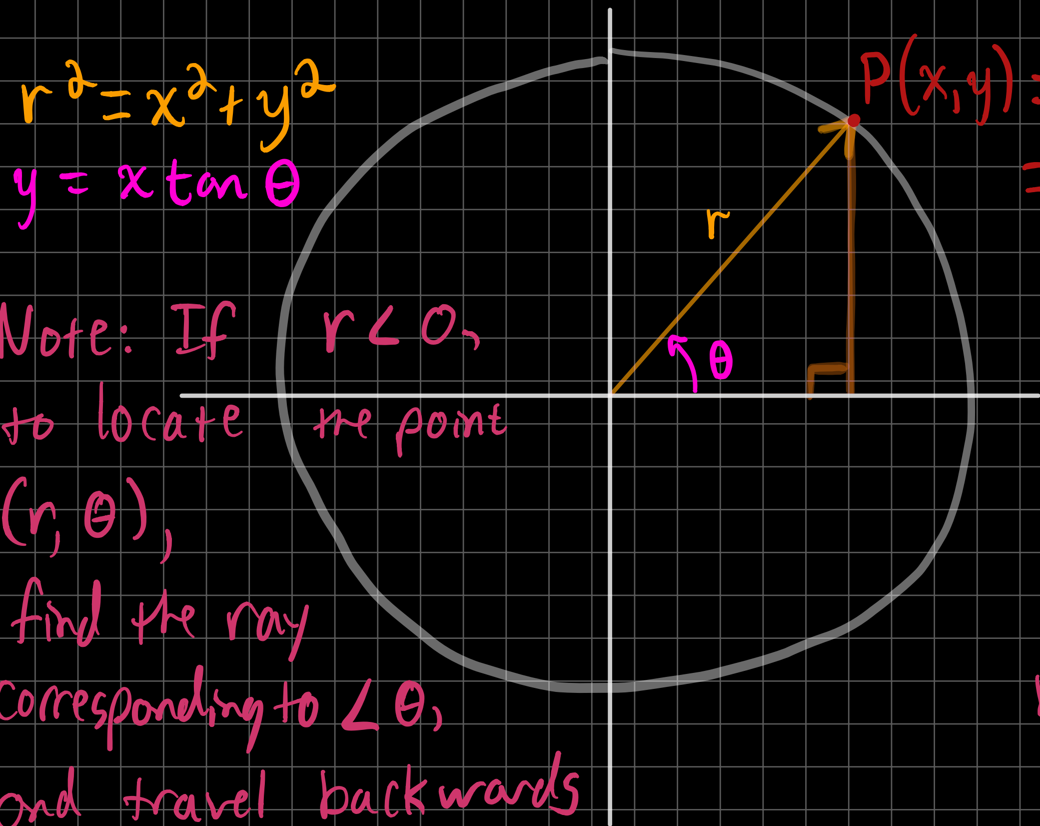
Double integrals in Polar Coordinates.

Review: Polar coordinates



$P(x,y) = (r\cos\theta, r\sin\theta) = (r,\theta)_p.$

I use a subscript P to indicate when an ordered pair gives radius & angle, rather than x & y values.

Note: If $r < 0$, to locate the point (r,θ), find the ray corresponding to $\angle\theta$, and travel backwards along the opposite ray ($\angle\theta+\pi$) a distance of $|r|$.

e.g. $P(1,1) = \left(\sqrt{2}, \frac{\pi}{4}\right)_p = \left(-\sqrt{2}, \frac{5\pi}{4}\right)_p = \left(\sqrt{2}, -\frac{7\pi}{4}\right)_p.$

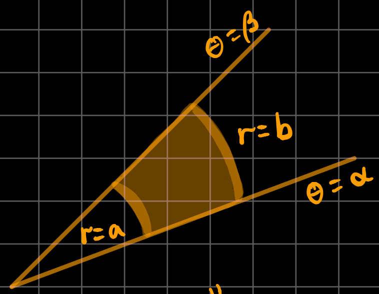
Though a given point has many polar representations, we often work with $r \geq 0$, $0 \leq \theta \leq 2\pi$.

Definition: A polar rectangle is a region of the form

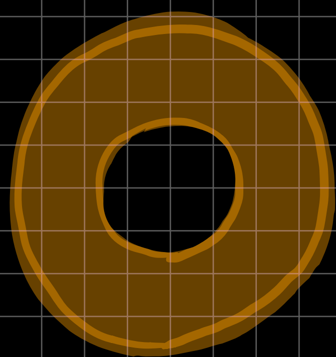
$$D = \{(r,\theta)_p : a \leq r \leq b,\ \alpha \leq \theta \leq \beta\}.$$

Here, we may stipulate $0 \le a \le b$ and $\alpha < \beta$, $\beta - \alpha \in [0, 2\pi]$.

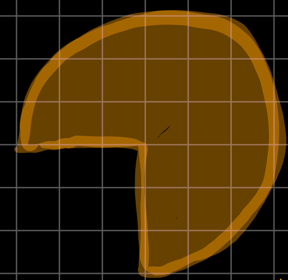
e.g. the following pictures illustrate some polar rectangles:



"The classic"



An Annulus



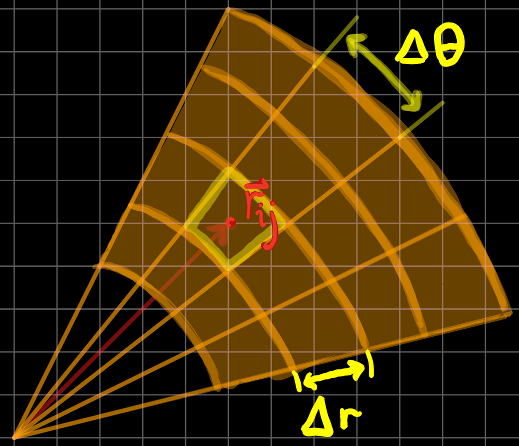
3/4 disk / tilted Pacman mid-bite.

To express a double integral over a polar rectangle D as an iterated integral, we need to understand how the Area differential dA transforms.

We thus build a Riemann sum approximation of $\iint_D f(x,y)\,dA$ where the sum is taken over an array of polar subrectangles of D:

$$\iint_D f(x,y)\,dA = \lim_{\substack{m \to \infty \\ n \to \infty}} \sum_{i=1}^{m} \sum_{j=1}^{n} f(\vec{r}_{ij}^{\,*})\,\Delta A_{ij}$$

where ΔA_{ij} is a polar rectangle area, and $\vec{r}_{ij}^{\,*}$ is a sample point in the ij^{th} polar subrectangle.



By the sector area formula:

$$A_{ij} = \frac{1}{2}\left(r_i^2 - r_{i-1}^2\right)\left(\theta_j - \theta_{j-1}\right)$$
$$= \left(\frac{r_{i-1} + r_i}{2}\right)(r_i - r_{i-1})(\theta_j - \theta_{j-1})$$
$$= \bar{r}_i \, \Delta r \, \Delta\theta$$

where $\bar{r}_i = \left(\frac{r_{i-1} + r_i}{2}\right)$ is the average radius in the ij^{th} polar subrectangle.

We can express $\vec{r}_{ij}^{\,*} = \langle x_{ij}^*, y_{ij}^* \rangle$ using polar coordinate variables as

$$\vec{r}_{ij}^{\,*} = \langle r_{ij}^* \cos\theta_{ij}^*, \; r_{ij}^* \sin\theta_{ij}^* \rangle$$

Where $(r_{ij}^* \, \theta_{ij}^*)_p = (x_{ij}^*, y_{ij}^*)$.

Then

$$\sum_{i=1}^{m}\sum_{j=1}^{n} f(\vec{r}_{ij}^{\,*}) \, \Delta A_{ij}$$
$$= \sum_{i=1}^{m}\sum_{j=1}^{n} f(r_{ij}^* \cos\theta_{ij}^*, \, r_{ij}^* \sin\theta_{ij}^*) \, \bar{r}_i \, \Delta r \, \Delta\theta$$

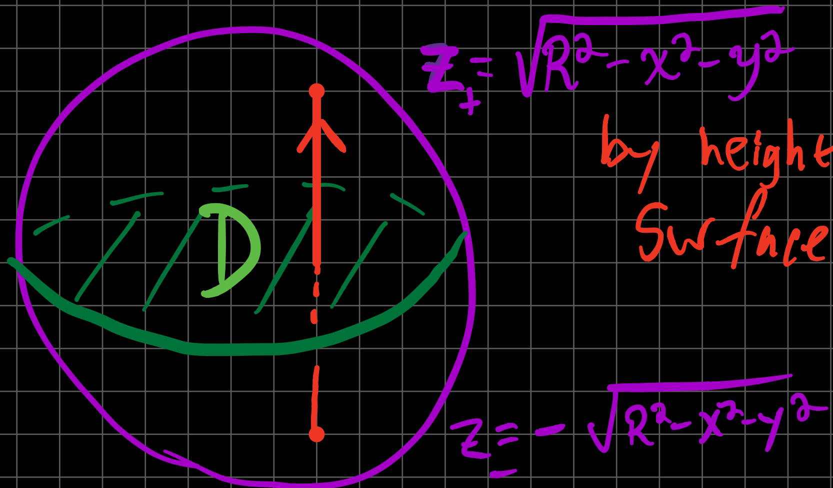
Choosing $r_{ij}^* = \bar{r}_i$, we recognize a Riemann sum relative to a rectangular region of $r\theta$ space, whence in the limit, by Fubini's Theorem (under appropriate assumptions on f):

$$\iint_D f(x,y)\,dA = \int_a^b \int_\alpha^\beta f(r\cos\theta, r\sin\theta)\,r\,d\theta\,dr.$$

Example: Use a double integral to calculate the volume of a sphere of radius R.

Solution: We first set up an integral in rectangular coordinates (to illustrate the impracticality of using rectangular coordinates for a perfectly round object). We will evaluate using polar coordinates.

Using an origin centered sphere $x^2+y^2+z^2=R^2$:



Integrand is determined by height between lower & upper surface: $z_+ - z_-$. Thus:

$$V = \iint_D z_+ - z_-\,dV$$

Since D is a radius R disk centered @ the origin:

$$D = \{(x,y) : -R \le x \le R, -\sqrt{R^2-x^2} \le y \le \sqrt{R^2-x^2}\}$$
$$= \{(r,\theta)_p : 0 \le r \le R, \ 0 \le \theta \le 2\pi\}$$

$$V = \int_{-R}^{R} \int_{-\sqrt{R^2-x^2}}^{\sqrt{R^2-x^2}} 2\sqrt{R^2-x^2-y^2}\, dy\, dx$$

by (†)

$$= \int_0^{2\pi} \int_0^R 2\sqrt{R^2-r^2}\, r\, dr\, d\theta$$

$2r\,dr = -d(R^2-r^2)$

$$= 2\pi \cdot \left(-\frac{2}{3}(R^2-r^2)^{3/2} \Big|_0^R \right)$$

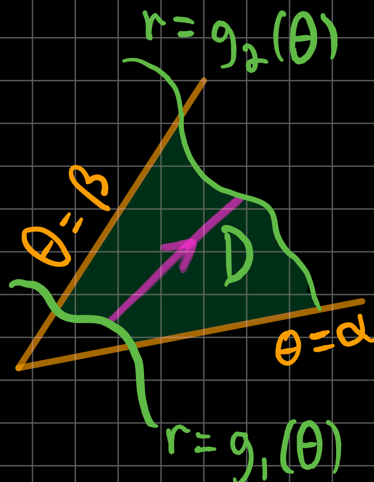
$$= \frac{4\pi}{3}\left(-0 - \left(-(R^2)^{3/2}\right)\right) = \frac{4}{3}\pi R^3 !!$$

Polar curves and general polar Regions:

Recall, a polar curve is a curve determined as all points $(r,\theta)_p$ such that r & θ satisfy a specified relation; often a polar curve is a graph of a function specified as $r = f(\theta)$.

The next Theorem expresses a double integral over a polar region with polar curve bounds as an iterated integral.

Theorem: Let g_1, g_2 be continuous functions defined on an interval $[\alpha, \beta]$, with $g_2(\theta) \geq g_1(\theta)$.



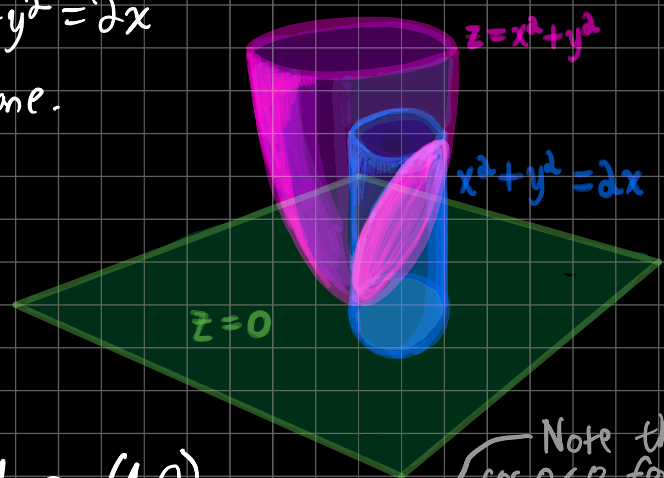
Let $D = \{(r,\theta)_p : g_1(\theta) \leq r \leq g_2(\theta),\ \alpha \leq \theta \leq \beta\}$.

If $f(x,y)$ is continuous on D, then

$$\iint_D f(x,y)\, dA = \int_\alpha^\beta \int_{g_1(\theta)}^{g_2(\theta)} f(r\cos\theta, r\sin\theta)\, r\, dr\, d\theta .$$

Example: Find the volume of the solid bounded above $z=0$, below $z=x^2+y^2$, and inside the cylinder $x^2+y^2=2x$.

Solution: The cylinder $x^2+y^2=2x$ bounds a disk D in the plane.



Indeed:

$$x^2+y^2=2x$$
$$\Updownarrow$$
$$x^2-2x+1+y^2=1$$
$$\Updownarrow$$
$$(x-1)^2+y^2=1$$

So D is a unit disk centered @ $(1,0)$.

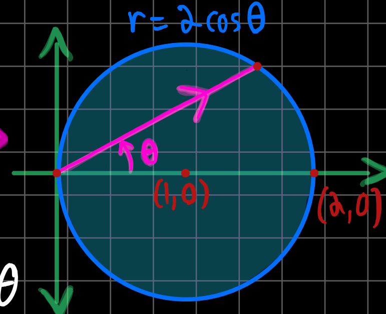
Expressing the boundary circle in polar coordinates:

$$\underbrace{x^2+y^2}_{= r^2} = \underbrace{2x}_{= 2r\cos\theta} \quad \rightsquigarrow \quad r = 2\cos\theta. \quad 0 \leq \theta \leq \pi$$

Note that $\cos\theta \leq 0$ for $\pi/2 \leq \theta \leq \pi$; convince yourself that $r=2\cos\theta$ double covers the disk for $0 \leq \theta \leq 2\pi$.

For a given $\theta \in [0, \pi]$, a point $P \in D$ along this θ ray has radius satisfying

$0 \le r \le 2\cos\theta$.



Thus $$V = \iint_D x^2+y^2 \, dA = \int_0^{\pi}\int_0^{2\cos\theta} r^2 \cdot r \, dr \, d\theta$$

$$= \int_0^{\pi} \frac{1}{4}(2\cos\theta)^4 \, d\theta = 4\int_0^{\pi} (\cos^2\theta)^2 \, d\theta$$

$$= 4\int_0^{\pi} \left(\frac{1+\cos 2\theta}{2}\right)^2 d\theta$$

$$= \int_0^{\pi} 1 + 2\cos 2\theta + \frac{1+\cos 4\theta}{2} \, d\theta$$

$$= \frac{3\pi}{2}$$

Here we use the double angle identity

$\cos(2\theta) = \cos^2\theta - \sin^2\theta$ (where $\sin^2\theta = 1-\cos^2\theta$)

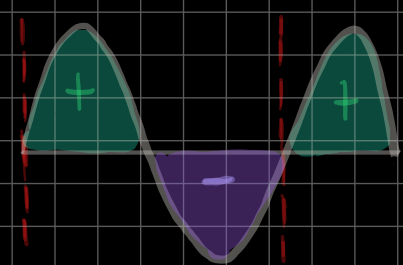
$= 2\cos^2\theta - 1$

$\Updownarrow$

$\cos^2\theta = \dfrac{1+\cos 2\theta}{2}$.

To handle these types of terms quickly, recall that

$$\int_0^{2\pi/b} \cos bt \, dt = 0 = \int_0^{2\pi/b} \sin bt \, dt.$$



Example: A polar curve of the form $r = \cos n\theta$ or $r = \sin n\theta$ for an integer n is called a "rose" (daisy would be a better name...)

a.) Sketch the rose $r = \cos 2\theta$.

b.) Find the area of one "petal" of this rose by setting up & evaluating a double integral.

a.) A useful fact: Recall, if $(x,y) = (r,\theta)_p$ is a point lying on a polar curve $r = f(\theta)$, then the slope of the tangent line to the polar curve through (x,y) is

$$\frac{dy}{dx} = \frac{\frac{d}{d\theta}(r\sin\theta)}{\frac{d}{d\theta}(r\cos\theta)} = \frac{\frac{dr}{d\theta}\sin\theta + r\cos\theta}{\frac{dr}{d\theta}\cos\theta - r\sin\theta};$$

and thus the slope of a line tangent @ $(0,0)$ to a curve through $(0,0)$ is $\frac{dy}{dx} = \tan\theta$, assuming $dr/d\theta \neq 0$ @ the origin. But, such a line necessarily coincides with a line making $\angle\theta$ with the x-axis!

Thus, to sketch a polar curve, it is quite useful to first determine for which θ the curve passes through the origin, i.e., to determine all θ such that $r(\theta) = f(\theta) = 0$.

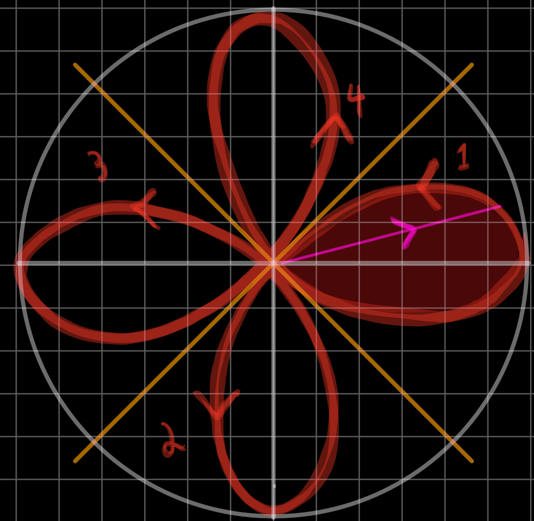
For $r = \cos(2\theta)$, we have $r = 0$ if and only if $\theta \in \{\frac{\pi}{4} + k\pi,\ k \text{ an integer}\}$. The curve is thus tangent to the lines $y = \pm x$ as it passes through $(0,0)$.

b.) The area is given by

$$A = \int_{-\frac{\pi}{4}}^{\frac{\pi}{4}} \int_0^{\cos(2\theta)} r \, dr \, d\theta$$

$$= \int_{-\frac{\pi}{4}}^{\pi/4} \frac{1}{2} \cos^2 2\theta \, d\theta$$

$$= \frac{\pi}{8}.$$



the 4-petaled rose $r = \cos 2\theta$.

A trick: By symmetry: $A = \frac{1}{4} \int_0^{2\pi} \int_0^{\cos(2\theta)} r \, dr \, d\theta$

$$= \frac{1}{8} \int_0^{2\pi} \cos^2 2\theta$$

$$= \frac{1}{16} \int_0^{2\pi} 1 + \cos 4\theta \, d\theta = \frac{\pi}{8}.$$