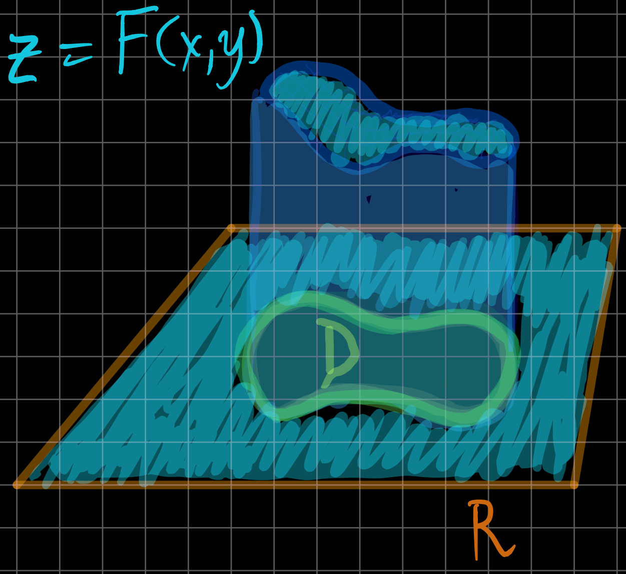
Double Integrals over General Regions

Assume $D \subset \mathbb{R}^2$ a compact (closed & bounded) region and $f(x,y)$ continuous over D.

Let R be a bounding rectangle of D.

Let $F(x,y) := \begin{cases} f(x,y) & \text{if } (x,y) \in D \\ 0 & \text{if } (x,y) \notin D \end{cases}$

Then Define $\iint_D f(x,y)\,dA := \iint_R F(x,y)\,dA$.

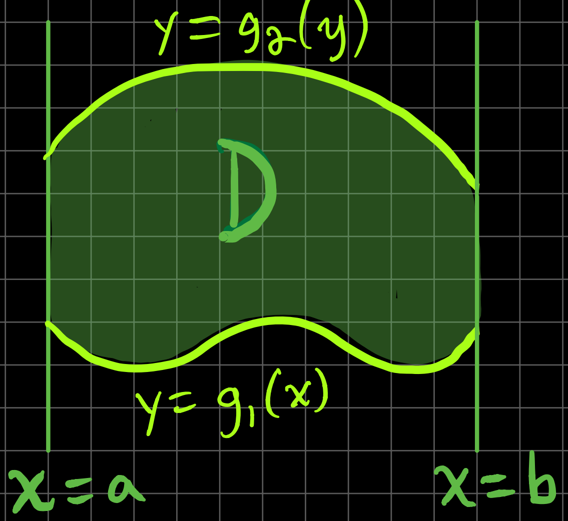


To compute such expressions via iterated integrals, we define two special, simple cases of Regions.

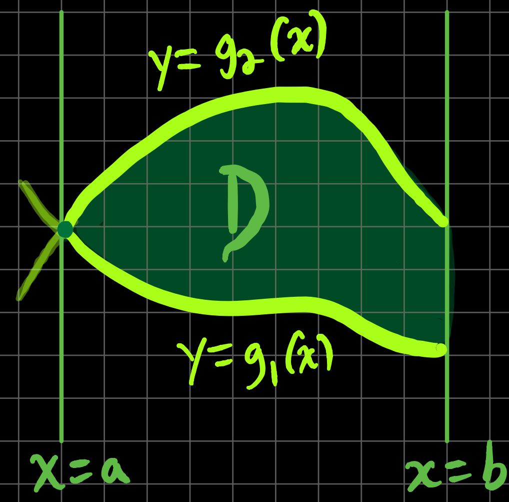
<u>Definition</u>: A region D is said to be of <u>type I</u> if there exists a pair of functions $g_1, g_2 : [a,b] \to \mathbb{R}$, such that

$$D = \{(x,y) : a \le x \le b,\ g_1(x) \le y \le g_2(x)\}.$$

That is, D admits a description as the set of all points between vertical lines $x=a$ & $x=b$, and lying above or on the graph of $y=g_1(x)$ but below or on $y=g_2(x)$.

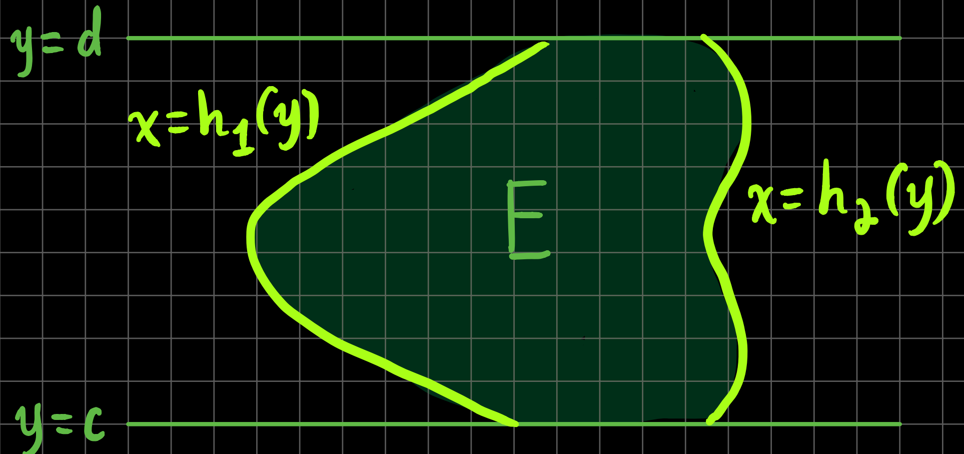


or

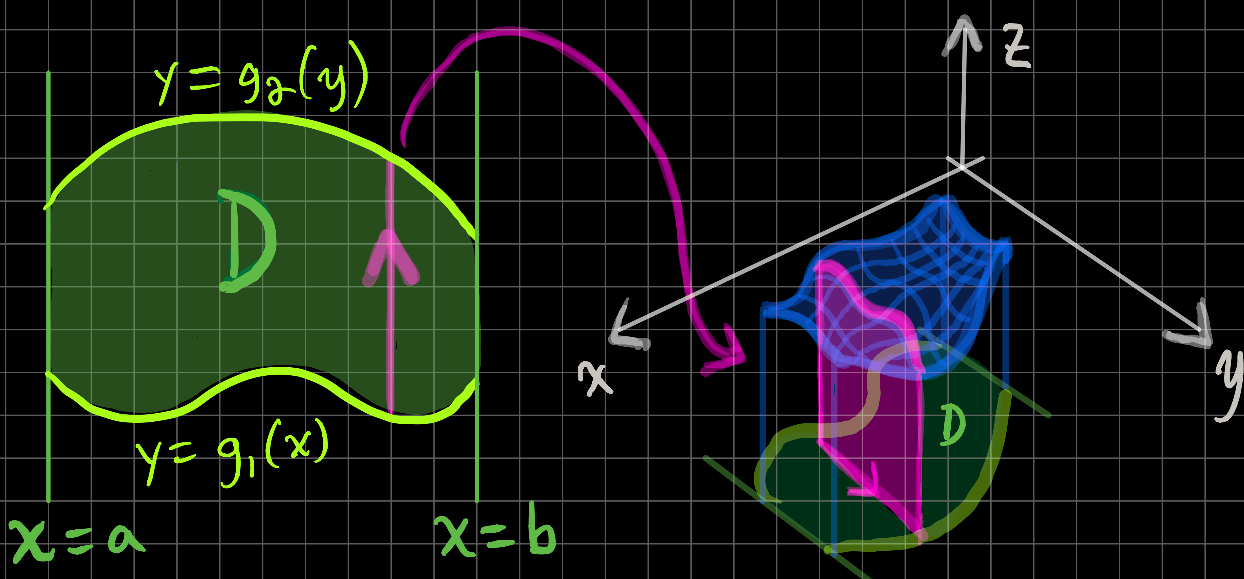


Similarly, we call $E \subset \mathbb{R}^2$ a <u>type II</u> region if there exists a pair of functions $h_1, h_2 : [c,d] \to \mathbb{R}$ such that

$$E = \{(x,y) : h_1(y) \leq x \leq h_2(y),\ c \leq y \leq d\}$$



For a type I region D, to calculate $\iint_D f(x,y)\,dA$ using the cross sectional area principle, one would take cross sections parallel to the y-axis, noting that the widths of the cross sections varies according to the functions $y = g_1(x)$ & $y = g_2(x)$.



$$A(x) = \int_{g_1(x)}^{g_2(x)} f(x,y)\,dy$$

<u>Proposition</u>: If $f(x,y)$ is integrable over a type I region

$$D = \{(x,y) : a \le x \le b,\ g_1(x) \le y \le g_2(x)\},$$

then

$$\iint_D f(x,y)\,dA = \int_{x=a}^{x=b} \int_{y=g_1(x)}^{y=g_2(x)} f(x,y)\,dy\,dx.$$

Similarly, if f is integrable over a type II region

$$E = \{(x,y) : h_1(y) \le x \le h_2(y),\ c \le y \le d\}$$

then

$$\iint_E f(x,y)\,dA = \int_{y=c}^{y=d} \int_{x=h_1(y)}^{x=h_2(y)} f(x,y)\,dx\,dy.$$

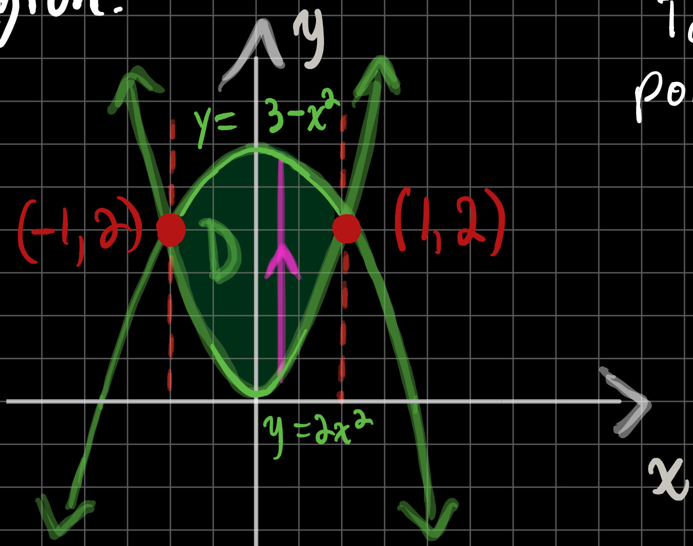
Think:

Type I bounds: $\displaystyle\int_{\text{left line}}^{\text{right line}} \int_{\text{bottom graph}}^{\text{top graph}}$

Type II bounds: $\displaystyle\int_{\text{bottom line}}^{\text{top line}} \int_{\text{left graph}}^{\text{right graph}}$

Example: Find the volume of the solid beneath $z = 6x^2y$ and above the region on the xy plane bounded by $y = 3 - x^2$ and $y = 2x^2$.

Step 1: Draw the region and determine bounds as either a type I or type II region.



To find intersection points:

$$3 - x^2 = y = 2x^2$$
$$\Updownarrow$$
$$3 = 3x^2$$
$$\Updownarrow$$
$$x = \pm 1,\ y = 2.$$

Region admits an easy type I description:

$$D = \{(x,y) : -1 \leq x \leq 1,\ 2x^2 \leq y \leq 3 - x^2\}.$$

Step 2: Set up the iterated integral:

$$V = \iint_D 6x^2y \, dA = \int_{-1}^{1} \int_{2x^2}^{3-x^2} 6x^2y \, dy \, dx.$$

Step 3: Compute!

$$V = \iint_D 6x^2y \, dA = \int_{-1}^{1} \int_{2x^2}^{3-x^2} 6x^2 y \, dy \, dx$$

$$= \int_{-1}^{1} 6x^2 \left[\tfrac{1}{2} y^2 \right]_{2x^2}^{3-x^2} dx$$

$$= \int_{-1}^{1} 3x^2(3-x^2)^2 - 3x^2(2x^2)^2 \, dx$$

$$= \int_{-1}^{1} 27x^2 - 18x^4 - 9x^6 \, dx$$

$$= 2\int_{0}^{1} 27x^2 - 18x^4 - 9x^6 \, dx$$

$$= 2\left[9x^3 - \tfrac{18}{5}x^5 - \tfrac{9}{7}x^7 \right]_0^1$$

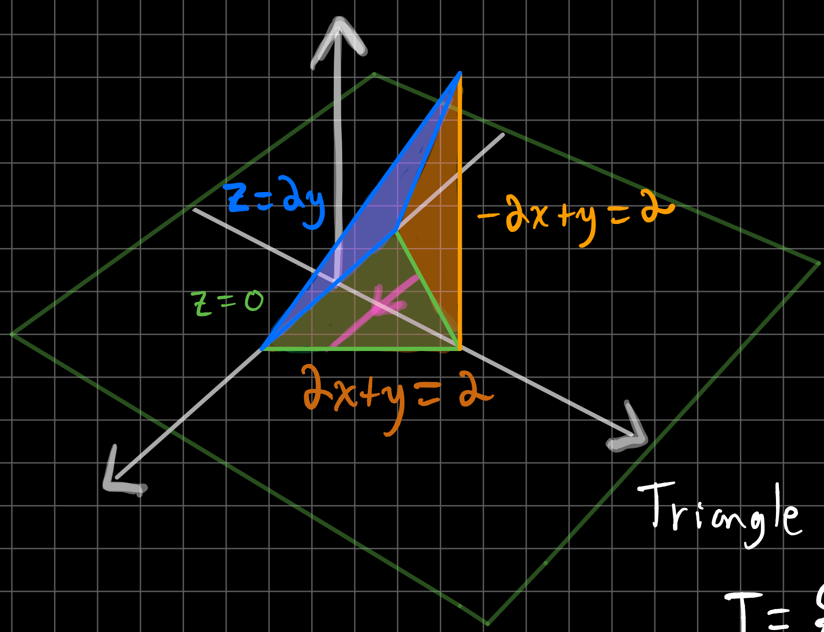
$$= \frac{288}{35}.$$

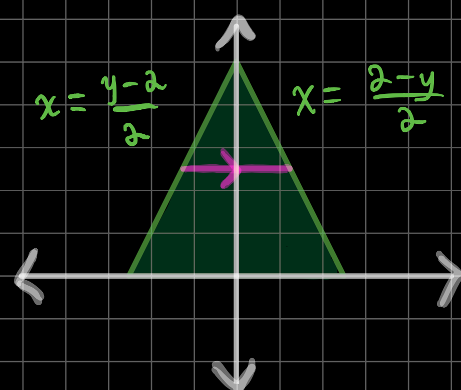
Can use that the integrand is an even polynomial, and the bounds are symmetric around 0:

$$\int_{-a}^{a} f(x)\,dx = \begin{cases} 2\int_0^a f(x)\,dx & \text{if } f(x) \text{ even} \\ 0 & \text{if } f(x) \text{ odd} \\ \int_{-a}^{a} f(x)\,dx & \text{if } f(x) \text{ neither even nor odd.} \end{cases}$$

Example: Find the volume of the solid bounded by the planes $z=0$, $z=2y$, $2x+y=2$, and $-2x+y=2$.

Solution: The solid is a tetrahedron, with base a triangle T in the xy plane bounded by $y=0$ (the line where the plane $z=2y$ meets the plane $z=0$), $y=2+2x$, and $y=2-2x$:





Triangle T admits a type $\underline{\text{II}}$ description:

$$T=\left\{(x,y): \frac{y-2}{2}\le x\le \frac{2-y}{2}, 0\le y\le 2\right\}.$$

Thus $$V=\iint_T 2y\,dA=\int_0^2\int_{\frac{y-2}{2}}^{\frac{2-y}{2}} 2y\,dx\,dy$$

$$=\int_0^2 \frac{1}{2}(2y-y^2-y^2+2y)\,dy$$

$$=\int_0^2 2y-y^2\,dy=\left[y^2-\frac{1}{3}y^3\right]_0^2=\frac{4}{3}.$$

Definition: A region E is called <u>elementary</u> if it can be expressed as both a type I region & a type II region.

Note: It can be advantageous to choose whether to represent an elementary region as a type I or type II region in order to select suitable bounds for iterated integrals, as our next example demonstrates.

Ex: Compute $\int_0^1 \int_x^1 \sin(y^2)\,dy\,dx$.

Question: Do you know an explicit antiderivative for $\sin(y^2)$? ...

Integrals like $S(x) = \int_0^x \sin(y^2)\,dy$ and

$$C(x) = \int_0^x \cos(y^2)\,dy$$

are called <u>Fresnel integrals.</u>

They admit no elementary description; they are <u>transcendental functions</u>.

So, instead of integrating in the given order, we will <u>reverse the order of integration</u>.

To reverse order of integration for an elementary region E:

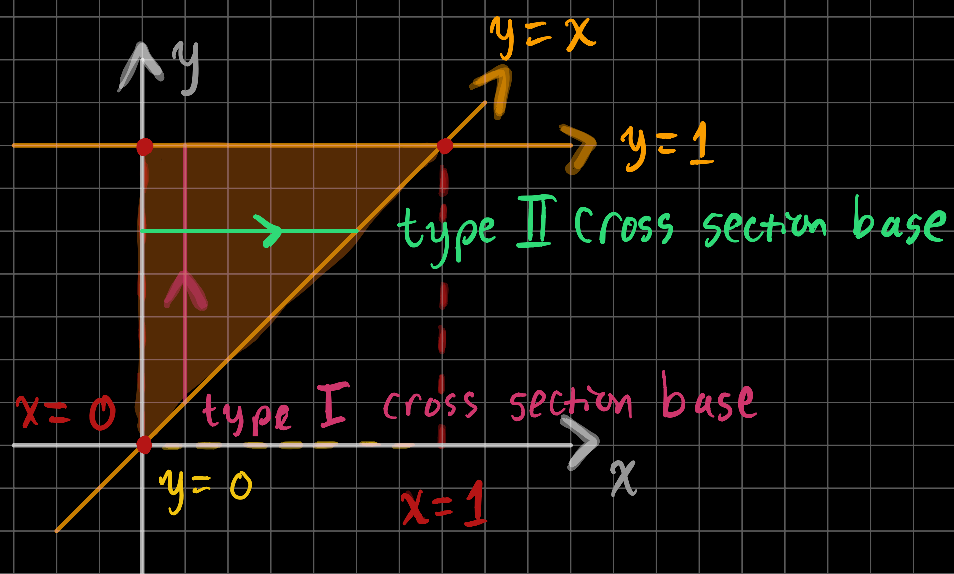
1.) Sketch E using the given description

2.) Inverting functions as necessary, describe the region as type II if given type I bounds, or as type I if given type II bounds.

3.) Use the new description to obtain new bounds.

For $\int_0^1 \int_x^1 \sin(y^2)\, dy\, dx$:

$$E = \{(x,y) : 0 \le x \le 1,\ x \le y \le 1\} \quad (\text{Type I})$$



$$E = \{(x,y) : 0 \le x \le y,\ 0 \le y \le 1\} \quad (\text{Type II})$$

Thus $\int_0^1 \int_x^1 \sin(y^2)\, dy\, dx = \int_0^1 \int_0^y \sin(y^2)\, dx\, dy.$

The latter iterated integral can readily be evaluated:

$$\int_0^1 \int_0^y \sin(y^2)\,dx\,dy = \int_0^1 x\sin(y^2)\Big|_0^y dy$$

$$= \int_0^1 y\sin(y^2)\,dy$$

$$= \frac{1}{2}\left(-\cos(y^2)\right)\Big|_0^1$$

$$= \frac{1}{2}\left(1-\cos(1)\right).$$

Properties of Double integrals

1.) Linearity (still): If f, g are each integrable over a general compact region $D \subset \mathbb{R}^2$ and $c \in \mathbb{R}$ is any constant, then

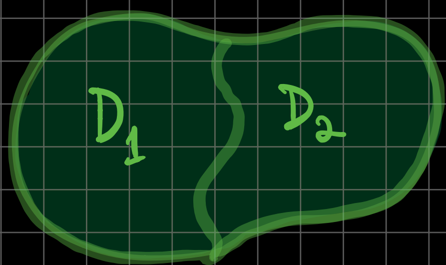
$$\iint_D cf(x,y)+g(x,y)\,dA = c\iint_D f(x,y)\,dA + \iint_D g(x,y)\,dA.$$

2.) If $f(x,y) \geq g(x,y)$ throughout D, then
$$\iint_D f(x,y)\,dA \geq \iint_D g(x,y)\,dA.$$

3.) $\iint_D 1 \, dA = A(D)$.

4.) If D can be written as a union of 2 disjoint sets D_1 & D_2, then

$$\iint_D f(x,y)\,dA = \iint_{D_1} f(x,y)\,dA + \iint_{D_2} f(x,y)\,dA.$$



$D = D_1 \cup D_2$.

5.) As before: suppose $f(x,y)$ is continuous over D, and let

$m = \min\limits_{(x,y)\in D} f(x,y)$, $M = \max\limits_{(x,y)\in D} f(x,y)$. Then:

$$m \cdot A(D) \le \iint_D f(x,y)\,dA \le M \cdot A(D),$$

where $A(D) = \text{area}(D)$.