MATH 131, Fall 2019
Quiz 11 Solutions

1. Let

$$f(x) = 6x^5 - \frac{1}{x} + \frac{1}{x^2} - \frac{1}{3\sqrt[3]{x^2}} + \frac{3}{1+x^2}\,.$$

Find a general antiderivative $F(x)$ of $f(x)$, valid wherever $f(x)$ is continuous.

Note that $f(x)$ is continuous for all real numbers except zero, and so in particular there are two connected intervals over which it is continuous: $(-\infty, 0)$ and $(0, \infty)$. Let I be a connected interval contained in one of these. Then using linearity, the power rule for anti-differentiation, together with knowledge of derivatives of logarithms and inverse trigonometric functions, we obtain a general antiderivative over I of the form

$$F(x) = x^6 - \ln|x| - \frac{1}{x} - \sqrt[3]{x} + 3\arctan(x) + C\,.$$

To obtain the most general antiderivative valid on $(-\infty, 0) \cup (0, \infty)$ we build a piecewise function

$$F(x) = \begin{cases} x^6 - \ln(x) - \frac{1}{x} - \sqrt[3]{x} + 3\arctan(x) + C_1 & \text{if } x > 0 \\ x^6 - \ln(-x) - \frac{1}{x} - \sqrt[3]{x} + 3\arctan(x) + C_2 & \text{if } x < 0 \end{cases},$$

where the different constants allow for different vertical shifts to the left and right of the discontinuity at $x = 0$.

2. Find the solution $f(x)$ of the following initial value problem: $f''(x) = e^x - \cos x$, $f'(0) = e$, $f(0) = 1$.

We anti-differentiate and apply the initial conditions given to determine the values of the constants at each stage:

$$f'(x) = e^x - \sin x + C \text{ and } f'(0) = e = e^0 - \sin(0) + C \implies C = e\,,$$

$$f(x) = e^x + \cos x + ex + D \text{ and } f(0) = 1 = e^0 + \cos(0) + e(0) + D \implies D = -1\,,$$

so the final function solving the initial value problem is

$$f(x) = e^x + \cos x + ex - 1\,.$$