MATH 131, Fall 2019
Quiz 10 Solutions

1. An open top box with a square base is to be made by cutting four squares away from the corners of a $4\,\text{ft} \times 4\,\text{ft}$ square piece of cardboard, and then folding up the sides. Find the maximum volume possible and the dimensions of the box achieving it.

If we cut away squares that are $x \times x$ square feet then the base of the box after folding up the sides is $4 - 2x$ feet long and equally wide, and the sides of the box are x feet tall, giving a volume formula $V(x) = x(4 - 2x)^2$ cubic feet.

We know that since the original piece of cardboard was $4\,\text{ft} \times 4\,\text{ft}$, the maximum value for x permissible is 2 feet, since this corresponds to decomposing the cardboard into four equal squares, all of which are removed, leaving a "box of volume zero" (Impractical? Yes! Impossible? No! A mathematician can imagine a 0-dimensional box...) The minimum value of x is zero feet, which corresponds to not removing any squares, not folding up the sides, and again gives a volume of zero (a completely flat box). Thus the range of permissible values for x is the closed interval $[0, 2]$, and by the continuity of $V(x)$ and the closed interval method, it suffices to compare values of the boundary cases $V(0) = 0 = V(2)$ to any critical values for critical numbers within the interval.

The critical numbers come from finding where the derivative of V with respect to x is 0. By the product and chain rules:

$$V'(x) = (4 - 2x)^2 - 4x(4 - 2x) = (4 - 2x - 4x)(4 - 2x) = 4(2 - 3x)(2 - x)\,,$$

and so the only critical numbers are those for which $V'(x) = 0$, which occurs if either $x = 2$ feet or $x = 2/3$ feet. Only $x = 2/3$ gives a nonzero volume, and this must yield a maximum. Thus the maximum possible volume is

$$V(2/3) = \frac{128}{27}\,\text{ft}^3\,,$$

which is obtained with dimensions $8/3\,\text{ft} \times 8/3\,\text{ft} \times 2/3\,\text{ft}$.