MATH 131, Fall 2019
Quiz 9 Solutions

1. Let $f(x) = \dfrac{x}{x^2+1}$.

(a) Find the intervals on which f is increasing, and the intervals on which f is decreasing.

$$f'(x) = \frac{(1)(x^2+1) - x(2x)}{(x^2+1)^2} = \frac{1-x^2}{(1+x^2)^2}\,.$$

Since $(1+x^2)^2 \geq 1$ for all real x, $f'(x) > 0 \iff 1-x^2 \geq 0 \iff 1 \geq x^2 \iff -1 \leq x \leq 1$. Observe that the critical numbers are $x = \pm 1$, and $f'(x) > 0$ on the interval $(-1,1)$ thus f is increasing on the interval $(-1,1)$. f is decreasing whenever $f'(x) < 0$, which occurs whenever $x^2 > 1$, or equivalently, for all x of absolute value larger than 1, and thus f is decreasing whenever x is in $(-\infty,-1) \cup (1,\infty)$.

(b) Find the intervals of concavity and any inflection points of f.

$$f''(x) = \frac{(-2x)(1+x^2)^2 - 2(1-x^2)(1+x^2)(2x)}{(1+x^2)^4} = \frac{2x^3-6x}{(1+x^2)^3}\,.$$

The sign of $f''(x)$ is determined solely by the sign of its numerator $2x^3 - 6x = 2x(x^2-3) = 2x(x-\sqrt{3})(x+\sqrt{3})$. If $x < -\sqrt{3}$, all three factors are negative, so $f''(x)$ is negative, while if $-\sqrt{3} < x < 0$ then the factor $x + \sqrt{3}$ becomes positive while the others remain negative, and so $f''(x)$ becomes positive. If $0 < x < \sqrt{3}$ then only the factor of $x - \sqrt{3}$ is negative, and the resulting product gives us a negative sign for $f''(x)$. Finally, if $x > \sqrt{3}$ all factors are positive and $f''(x)$ is also positive. Thus, by the concavity test f is concave down on $(-\infty,-\sqrt{3}) \cup (0\sqrt{3})$ and concave up on $(-\sqrt{3},0) \cup (\sqrt{3},\infty)$. Since the concavity changes at $x = -\sqrt{3}$, $x = 0$ and $x = \sqrt{3}$, there are points of inflection at $(-\sqrt{3},-\sqrt{3}/4)$, $(0,0)$, and $9\sqrt{3},\sqrt{3}/4)$.

(c) Find and classify all local extrema using either the first or second derivative tests.

Since the derivative is defined for all real x, it suffices to check and classify the critical values associated to the two critical numbers $x = -1$ and $x = 1$. By the first derivative test, since $f'(x)$ changes sign from negative to positive as x crosses -1, we deduce that $\big(-1, f(-1)\big) = (-1,-1/2)$ is a local minimum. On the other hand, as x crosses 1 the sign of f' changes from positive to negative, whence $\big(1, f(1)\big) = (1, 1/2)$ is a local maximum. If one prefers the second derivative test, it suffices to use that f is concave up when $x = -1$ to deduce that $(-1, -1/2)$ is a local minimum and that f is concave down when $x = 1$ to deduce that $(1, 1/2)$ is a local maximum.

(d) Determine any asymptotes of f.

Note that $f(x)$ is defined for all real numbers and thus there are no finite values a such that $\lim_{x\to a} f(x)$, $\lim_{x\to a^-} f(x)$, or $\lim_{x\to a^+} f(x)$ are infinite. Thus f has no vertical asymptotes. However, f does have a horizontal asymptote:

$$\lim_{x\to\infty} \frac{x}{x^2+1} = \lim_{x\to\infty} \frac{1}{x+1/x} = 0\,,$$

$$\lim_{x\to-\infty} \frac{x}{x^2+1} = \lim_{x\to-\infty} \frac{1}{x+1/x} = 0\,,$$

whence $y = 0$ is a horizontal asymptote of $y = f(x)$.

(e) Use the information gathered in (a)-(d) to sketch a graph of f.

See figure 1.

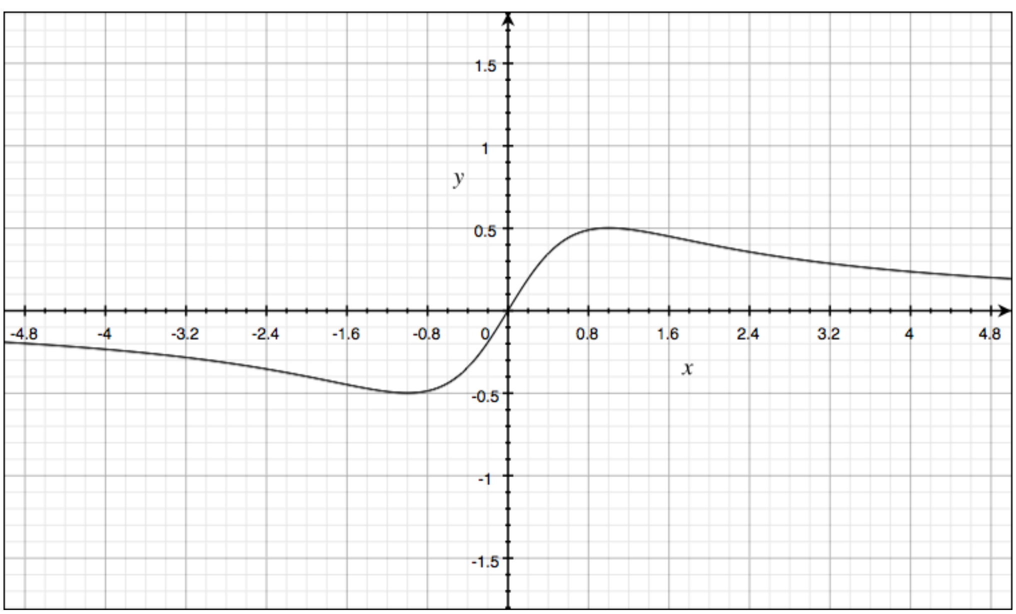

Figure 1: The graph of $y = \frac{x}{x^2+1}$.

2. Let $g(x) = x^2 - 3x^{2/3}$.

(a) Find the intervals on which g is increasing, and the intervals on which g is decreasing.

$$g'(x) = 2x - 2x^{-1/3} = \frac{2}{\sqrt[3]{x}}(x^{4/3} - 1)\,.$$

Observe that the critical numbers are $x = \pm 1$ and $x = 0$, since $(\pm 1)^{4/3} - 1 = 0$ and there are no other real roots of $x^{4/3} - 1$, while $\frac{2}{\sqrt[3]{x}} \neq 0$ for all nonzero x but fails to exist at $x = 0$.Note also that the numerator is positive for x between -1 and 1, and negative otherwise, while the denominator has the same sign as x itself, whence the function g is increasing when x is in $(-1, 0) \cap (1, \infty)$ and decreasing when x is in $(-\infty, -1) \cap (0, 1)$.

(b) Find the intervals of concavity and any inflection points of g.

$$g''(x) = 2 + \frac{2}{3}x^{-4/3} > 0 \text{ for all } x \neq 0\,.$$

Thus the function $g(x)$ is concave up on $\mathbb{R} - \{0\} = (-\infty, 0) \cup (0, \infty)$.

(c) Find and classify all local extrema using either the first or second derivative tests.

By the second derivative test, since $g''(\pm 1) = \frac{8}{3} > 0$, the critical points $(-1, -2)$ and $(1, -2)$ are both local minima. The second derivative fails to exist at $x = 0$, as does the first derivative, however $g(0)$ is defined and equal to 0, and we note that the first derivative goes from positive to negative across $x = 0$, and thus by the first derivative test $g(x)$ has a local maximum at $(0, 0)$. Further, by considering limits $\lim_{x\to 0^-} g(x)$ and $\lim_{x\to 0^+} g(x)$, one can show that there is a vertical tangent line to $g(x)$ at $(0, 0)$, whence this is a *cusp local maximum* .

(d) Determine any asymptotes of g.

The function $g(x)$ is continuous for all real numbers and has no vertical asymptotes. Observe that $\lim_{x\to\infty} g(x) = \lim_{x\to\infty} x^{2/3}(x^{4/3} - 3) = \infty$, and similarly $\lim_{x\to-\infty} g(x) = \infty$ whence there are no horizontal asymptotes.

(e) Use the information gathered in (a)-(d) to sketch a graph of g.

See figure 2.

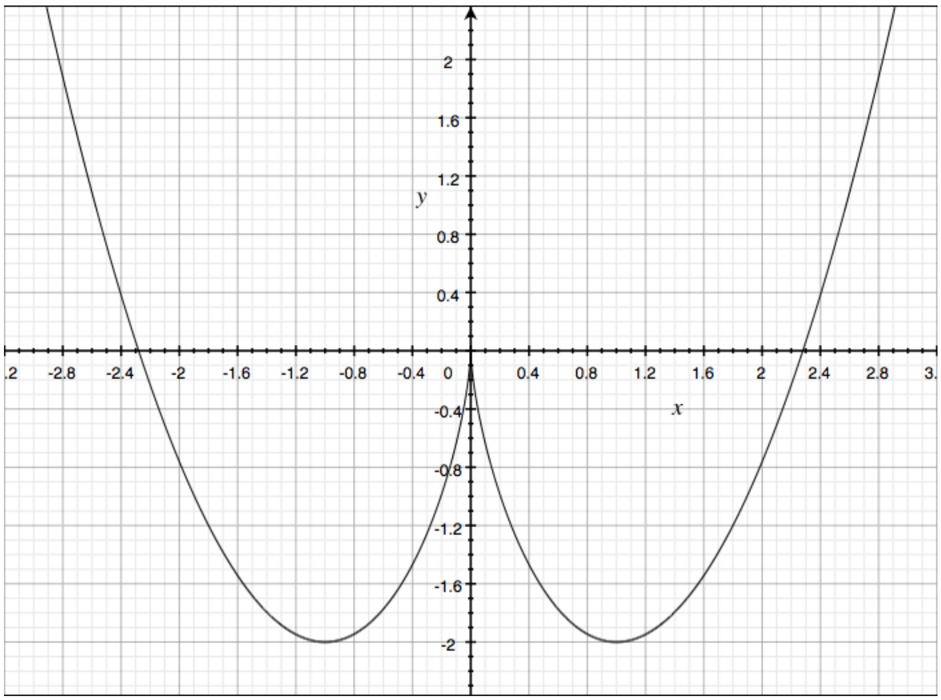

Figure 2: The graph of $y = x^2 - 3x^{2/3}$.