MATH 131, Fall 2019
Quiz 8 Solutions

1. Show that for any real numbers a and b,

$$|\sin(2b) - \sin(2a)| \leq 2|b-a|\,.$$

Let a and b be distinct real numbers. The trigonometric function $\sin x$ is continuous and differentiable for all real numbers x, and thus it is continuous on the closed interval with endpoints a and b, and differentiable on the open interval with endpoints a and b, with derivative $\frac{\mathrm{d}}{\mathrm{d}x}\sin(2x) = 2\cos(2x)$. Thus by the mean value theorem, there is a number c between a and b such that[1]

$$\sin(2b) - \sin(2a) = 2\cos(2c)(b-a)\,.$$

Since $0 \leq |\cos(2c)| \leq 1$ for any real c, we can take absolute values on both sides of our equation to conclude:

$$|\sin(2b) - \sin(2a)| = |2\cos(2c)||b-a| \leq 2|b-a|\,.$$

Of course, if $a = b$ we cannot apply the MVT as the interval between a and b consists of a single point $x = a = b$. In this case however, both sides of the desired inequality must yield 0, and so the inequality is still true when $a = b$. Thus, for any real numbers a and b, we conclude $|\sin(2b) - \sin(2a)| \leq 2|b-a|$, as was to be shown.

[1] Note that although in the MVT hypotheses one assumes $a < b$, it does not matter if instead $b < a$: switching the labels a and b and then multiplying the equation $f(a) - f(b) = f'(c)(a-b)$ by -1 on both sides recovers the MVT conclusion $f(b) - f(a) = f'(c)(b-a)$ in its original symbolic form.