MATH 131, Fall 2019
Quiz 8 Solutions

his quiz has two sides.

1. Show that for any real numbers a and b,

$$|\cos b - \cos a| \leq |b - a|\,.$$

Let a and b be distinct real numbers. The trigonometric function $\cos x$ is continuous and differentiable for all real numbers x, and thus it is continuous on the closed interval with endpoints a and b, and differentiable on the open interval with endpoints a and b, with derivative $\frac{\mathrm{d}}{\mathrm{d}x}\cos(x) = -\sin(x)$. Thus by the mean value theorem, there is a number c between a and b such that[1]

$$\cos(b) - \cos(a) = -\sin(c)(b - a)\,.$$

Since $0 \leq |\sin(c)| \leq 1$ for any real c, we can take absolute values on both sides of our equation to conclude:

$$|\cos(b) - \cos(a)| = |-\sin(c)||b - a| \leq |b - a|\,.$$

Of course, if $a = b$ we cannot apply the MVT as the interval between a and b consists of a single point $x = a = b$. In this case however, both sides of the desired inequality must yield 0, and so the inequality is still true when $a = b$. Thus, for any real numbers a and b, we conclude $|\cos b - \cos a| \leq |b - a|$, as was to be shown.

[1] Note that although in the MVT hypotheses one assumes $a < b$, it does not matter if instead $b < a$: switching the labels a and b and then multiplying the equation $f(a) - f(b) = f'(c)(a - b)$ by -1 on both sides recovers the MVT conclusion $f(b) - f(a) = f'(c)(b - a)$ in its original symbolic form.

2. Show that $f(x) = 3x - e^{\cos x}$ has a unique real root.

Note that $f(x)$ is continuous and differentiable for all real numbers, as it consists of a difference of a linear function with a composition of a trigonometric function with an exponential function, each class of which is continuous and differentiable on their domains (in this case all share the domain of the entire real line $\mathbb{R}$, and differences of continuous and differentiable functions remain continuous and differentiable. In particular, f is continuous on any closed interval, and also differentiable on any open interval.

Observe that $f(0) = -e < 0$, and $f(\pi/2) = 3\pi/2 - e^0 = 3\pi/2 - 1 > 0$. By continuity and the intermediate value theorem, since $f(0) = -e < 0 < 3\pi/2 - 1 = f(\pi/2)$, we conclude that there is a real number r, $0 < r < \pi/2$ such that $f(r) = 0$, and thus f has a real root.

Now, suppose f has more than one distinct real root. Let a and b be two such roots. Note that distinct means precisely that $a \neq b$, but we may assume without loss of generality that $a < b$. We know that f is continuous on the interval $[a, b]$, and differentiable on the interval (a, b). Now, since a and b are both roots, $f(a) = 0 = f(b)$, and so under the assumption of $a < b$ being distinct roots of f, there is an interval $[a, b]$ over which f meets the hypotheses of Rolle's theorem, and one concludes that between these roots there must be a number c with $f'(c) = 0$.

However, upon differentiation of f, we have

$$f'(x) = 3 + \sin(x)e^{\cos x}\,.$$

Since $0 \leq |\sin x| \leq 1$ and $0 \leq |\cos x| \leq 1$, we know that $|\sin(x)e^{\cos x}| \leq |1|e^1 = e$. Since $3 > e$, we conclude

$$f'(x) > 0 \text{ for all } x \text{ in } \mathbb{R}\,.$$

Indeed, the smallest possible value of $f'(x)$ is

$$f'(3\pi/2) = 3 - e > 0\,.$$

This contradicts the existence of a c such that $f'(c) = 0$. Thus the assumption of at least two distinct roots is untenable, and we conclude that the root r guaranteed by the intermediate value theorem is in fact the unique real root of this function.

3. Show that for all $x \geq 1$ the functions $f(x) = 2\cot^{-1}\left(\frac{1}{\sqrt{x}}\right)$ and $g(x) = \cos^{-1}\left(\frac{2\sqrt{x}}{1+x}\right)$ differ by a constant, and determine the value of that constant.

First, we compute the derivatives of these functions, using the assumption that $x > 1$ as needed during the simplification:

$$\begin{aligned} f'(x) &= 2\frac{(-1)}{1+(1/\sqrt{x})^2} \cdot \frac{\mathrm{d}}{\mathrm{d}x}\left(\frac{1}{\sqrt{x}}\right) \\ &= \frac{-2}{1+1/x} \cdot \frac{-1}{2x\sqrt{x}} \\ &= \frac{1}{x\sqrt{x}+\sqrt{x}} = \frac{1}{\sqrt{x}(x+1)} \,. \end{aligned}$$

$$\begin{aligned} g'(x) &= \frac{-1}{\sqrt{1-(2\sqrt{x}/(1+x))^2}} \cdot \frac{\mathrm{d}}{\mathrm{d}x}\left(\frac{2\sqrt{x}}{1+x}\right) \\ &= \frac{-1}{\sqrt{1-4x/(1+x)^2}} \cdot \left(\frac{\frac{1+x}{\sqrt{x}} - 2\sqrt{x}}{(1+x)^2}\right) \\ &= \frac{-(1+x)}{\sqrt{1+2x+x^2-4x}} \cdot \frac{1-x}{\sqrt{x}(1+x)^2} \\ &= \frac{-(1+x)}{\sqrt{(x-1)^2}} \cdot \frac{1-x}{\sqrt{x}(1+x)^2} \\ &= \frac{1+x}{1-x} \cdot \frac{1-x}{\sqrt{x}(1+x)^2} && \text{since } x > 1 \implies -\sqrt{(x-1)^2} = 1-x \\ &= \frac{1}{\sqrt{x}(x+1)} \,. \end{aligned}$$

Since $f'(x) = g'(x)$ for $x > 1$, we conclude that the functions differ by a constant, i.e., that there is some value C such that $f(x) = g(x) + C$. To find C, we evaluate the two functions at a value, such as $x = 1$:

$$f(1) = 2\cot^{-1}(1/1) = \frac{\pi}{2}\,, \quad g(1) = \cos^{-1}(2/2) = 0\,,$$

thus taking $C = \pi/2$, we have that

$$2\cot^{-1}\left(\frac{1}{\sqrt{x}}\right) = \cos^{-1}\left(\frac{2\sqrt{x}}{1+x}\right) + \frac{\pi}{2}\,.$$

4. A point $x = a$ is called a *fixed point*[2] of a function $f(x)$ if $f(a) = a$. Show that if $f'(x) \neq 1$ for all x then f has at most one fixed point.

Define a new function $g(x) = f(x) - x$. Note that a zero $x = a$ of $g(x)$ corresponds to a fixed point of $f(x)$: $0 = g(a) \iff 0 = f(a) - a \iff f(a) = a$. Now, suppose that $f(x)$ has two distinct fixed points, a and b, and without loss of generality, suppose $a < b$. Then assuming $f'(x)$ exists for all real numbers, we know that $f(x)$ is continuous for all real numbers (since differentiability implies continuity), and thus f is continuous on the interval $[a, b]$ and differentiable on its interior (a, b). We conclude that $g(x)$, being a difference of f and a linear function is also continuous on the interval $[a, b]$ and differentiable on its interior (a, b). Moreover, since a and b are fixed points of f, $g(a) = 0 = g(b)$. Thus, applying Rolle's theorem to g, we conclude that there is a value c in the interval (a, b) such that $g'(c) = 0$. But then $g'(c) = f'(c) - 1 = 0 \implies f'(c) = 1$. If $f'(x) \neq 1$ for any real x, then no such c can exists, so such a function thus cannot have two distinct fixed points, and therefore has at most one fixed point.

[2]The terminology "fixed point" is evocative if one considers the effect of *iterating* the map $x \mapsto f(x)$: if f is a map from a domain $D \subseteq \mathbb{R}$ with range contained in D, then consider for any number c in the domain D of f the sequence $\mathcal{O}(f, c) := \{c, f(c), f(f(c)), f(f(f(c))), \ldots, f^n(c), \ldots\}$, called the *orbit of c under the map $x \mapsto f(x)$* . Then a fixed point $x = a$ is one for which this sequence is constant: repeated applications of f leave a fixed, and a has *constant orbit'* under f.