MATH 131, Fall 2019
Quiz 7 Solutions

1. A particle moves along a line according to the position function $s(t) = (t^2 - 4t)^2$, where s is in meters and t is in seconds.

(a) Find the particle's maximum and minimum displacements from its original position in the first five seconds, and indicate all times t, $0 \leq t \leq 5$, where it achieves these displacements.

Since the position function is a polynomial, it is continuous and differentiable everywhere, and in particular it is continuous on the closed interval $[0, 5]$ and differentiable on $(0, 5)$, so we can apply the closed interval method:

1. Find critical values of $s(t)$ in $(0, 5)$, using Fermat's theorem to find critical numbers,
2. Compute $s(0)$ and $s(5)$,
3. Compare the values: the largest will be the maximum displacement in the first five seconds of motion, and the minimum value will give the minimum displacement.

Fermat's theorem says that if the function's derivative exists at a local extremum, then the derivative is 0 at the critical number giving the extremum. Thus, to find critical values we first find the critical times t_c such that the velocity $v(t_c) = s'(t_c) = 0$. We may differentiate by chain rule:

$$v(t) = \frac{\mathrm{d}}{\mathrm{d}t}(t^2 - 4t)^2 = 2(t^2 - 4t) \cdot (2t - 4) = 4t(t - 2)(t - 4)\,.$$

The zeroes of $v(t)$ happen at 0, 2 and 4 seconds, all of which are within the first 5 seconds of motion. The critical values are

$$s(0) = 0\,\text{meters}\,, \quad s(2) = 16\,\text{meters}\,, \text{ and } \quad s(4) = 0\,\text{meters}.$$

Checking the other boundary, we have $s(5) = 25$ meters. The maximum displacement is thus 25 meters occurring 5 seconds into the motion, and the minimum displacement is thus 0 meters, occurring 0 seconds into the motion and also 4 seconds into the motion.

(b) Find the *total distance* traveled by the particle in the first five seconds.

The particle is moving forward when the velocity is positive, and backwards when the velocity is negative. Note that

$$v(t) = 4t(t - 2)(t - 4) > 0 \text{ if } 0 < t < 2\,, \text{ or } 4 < t\,,$$

$$v(t) = 4t(t - 2)(t - 4) < 0 \text{ if } 2 < t < 4\,.$$

The particle is at rest at the critical times $t = 0$, 2, and 4 seconds. Thus, on the interval $[0, 5]$, the particle moves forward for times t in $(0, 2) \cup (4, 5)$ and moves backwards for times t in $(0, 2)$, whence the total distance traveled in the first five seconds of motion is

$$\begin{aligned} D &= |s(2) - s(0)| + |s(4) - s(2)| + |s(5) - s(4)| \\ &= |16\,\mathrm{m} - 0\,\mathrm{m}| + |0\,\mathrm{m} - 16\,\mathrm{m}| + |25\,\mathrm{m} - 0\,\mathrm{m}| \\ &= 16\,\mathrm{m} + 16\,\mathrm{m} + 25\,\mathrm{m} = 57\,\mathrm{m}\,. \end{aligned}$$