MATH 131, Fall 2019
Quiz 7 Solutions

1. A particle moves along a line according to the position function $s(t) = t^4 - 6t^3 + 10t^2 - 6t + 4$, where s is in meters and t is in seconds.

(a) Find the particle's maximum and minimum displacements from its original position in the first three seconds, and indicate all times t, $0 \leq t \leq 3$, where it achieves these displacements.

Since the position function is a polynomial, it is continuous and differentiable everywhere, and in particular it is continuous on the closed interval $[0, 3]$ and differentiable on $(0, 3)$, so we can apply the closed interval method:

1. Find critical values of $s(t)$ in $(0, 3)$, using Fermat's theorem to find critical numbers,
2. Compute $s(0)$ and $s(3)$,
3. Compare the values: the largest will be the maximum displacement in the first five seconds of motion, and the minimum value will give the minimum displacement.

Fermat's theorem says that if the function's derivative exists at a local extremum, then the derivative is 0 at the critical number giving the extremum. Thus, to find critical values we first find the critical times t_c such that the velocity $v(t_c) = s'(t_c) = 0$. Differentiating:

$$v(t) = \frac{\mathrm{d}}{\mathrm{d}t}\left(t^4 - 6t^3 + 10t^2 - 6t + 4\right) = 4t^3 - 18t^2 + 20t - 6\,.$$

It's may not be immediately obvious that this factors, but it does: observe that $4 - 18 + 20 - 6 = 0$, whence $t - 1$ divides the polynomial. Thus

$$4t^3 - 18t^2 + 20t - 6 = 2(2t^3 - 9t^2 + 10t - 3) = 2(t-1)(2t^2 - 7t + 3) = 2(t-1)(2t-1)(t-3)\,.$$

Thus, $v(t) = 2(t-1)(2t-1)(t-3)$, and the critical times are $t_c = 1/2$, 1, and 3 seconds. The corresponding critical values are

$$s(1/2) = 45/16\,\text{meters} = 2.8125\,\text{meters}\,, \quad s(1) = 3\ \text{meter}\,, \text{ and } \quad s(3) = -5\,\text{meters}\,.$$

One of the critical times is a boundary point, but we still must calculate the function's value at the other boundary point: $s(0) = 4\,\text{meters}$. The smallest value among these is $s(3) = -5\,\text{meters}$, and the largest value is $s(0) = 4\,\text{meters}$. Thus the minimum displacement during the first three seconds of motion is -5 meters occurring at 3 seconds, and the maximum displacement during the first three seconds of motion is 4 meters, occurring at 0 seconds. See the endnote about polynomial evaluation and factoring to learn about a slick way to do a problem like this without a calculator. Of course, numbers given on in-class problems will be chosen to minimize the arithmetic and algebraic difficulties, to a point.

(b) Find the *total distance* traveled by the particle in the first three seconds.

The particle is moving forward when the velocity is positive, and backwards when the velocity is negative. Note that

$$v(t) = 2(t-1)(2t-1)(t-3) > 0 \text{ if } 1/2 < t < 1\,,\ \text{or } 3 < t\,,$$

$$v(t) = 2(t-1)(2t-1)(t-3) < 0 \text{ if } t < 1/2\,,\ \text{or } 1 < t < 3\,.$$

The particle is at rest at the critical times $t = 1/2$, 1, and 3 seconds. Thus, on the interval $[0,3]$, the particle moves backwards for times t in $(0,1/2) \cup (1,3)$ and moves forward for times t in $(1/2,1)$, whence the total distance traveled in the first five seconds of motion is

$$\begin{aligned} D &= |s(1/2) - s(0)| + |s(1) - s(1/2)| + |s(3) - s(1)| \\ &= |45/16\,\mathrm{m} - 4\,\mathrm{m}| + |3\,\mathrm{m} - 4\,\mathrm{m}| + |-5\,\mathrm{m} - 5\,\mathrm{m}| \\ &= 19/16\,\mathrm{m} + 1\,\mathrm{m} + 9\,\mathrm{m} = 179/16\,\mathrm{m} = 11.1875\,\text{meters}\,. \end{aligned}$$

Endnote on Polynomial Evaluation and Division. To more easily calculate values such as $s(1/2)$, it can be helpful to rewrite the polynomial as follows:

$$\begin{aligned} s(t) &= t^4 - 6t^3 + 10t^2 - 6t + 4 = t(t^3 - 6t^2 + 10t - 6) + 4 \\ &= t\Big(t(t^2 - 6t + 10) - 6\Big) + 4 \\ &= t\Big(t\big(t(t-6) + 10\big) - 6\Big) + 4\,. \end{aligned}$$

For example, to find $s(3)$:

$$\begin{aligned} s(3) &= 3\Big(3\big(3(3-6) + 10\big) - 6\Big) + 4 \\ &= 3\Big(3\big(3(-3) + 10\big) - 6\Big) + 4 = 3\Big(3\big(-9 + 10\big) - 6\Big) + 4 \\ &= 3\big(3(1) - 6\big) + 4 = 3(-3) + 4 \\ &= 9 - 4 = 5\,. \end{aligned}$$

Evaluating using this form is referred to as *Horner's method*, and is equivalent to synthetic division and the polynomial remainder theorem used together to evaluate the polynomial. One can write it in the latter way as follows: For a polynomial $p(x) = a_n x^n + a_{n-1} x^n - 1 + \ldots + a_1 x + a_0$, to evaluate $p(c)$,

1. list the coefficients of the polynomial, including coefficients of 0 for any terms of degrees not written:

$$\begin{array}{|ccccc} a_n & a_{n-1} & \cdots & a_1 & a_0 \\ & & & & \\ \hline \end{array}$$

2. place the value c at which you wish to evaluate $p(x)$ to the left of the listed coefficients

$$\begin{array}{c|ccccc} & a_n & a_{n-1} & \cdots & a_1 & a_0 \\ c & & & & & \\ \hline \end{array}$$

3. bring down the first coefficient, a_n, to the bottom row, and then multiply by the value c, and place this product beneath the second coefficient:

$$
c\;\begin{array}{|c|cccc}
a_n & a_{n-1} & \dots & a_1 & a_0 \\
 & c\cdot a_n & & & \\
\hline
a_n & & & &
\end{array}
$$

4. add the second column and place the result in the last row, in the second column:

$$
c\;\begin{array}{|c|c|c|cc}
a_n & a_{n-1} & \dots & a_1 & a_0 \\
 & a_n\cdot c & & & \\
\hline
a_n & a_{n-1}+a_n\cdot c & & &
\end{array}
$$

5. repeat the process of multiplying entries of the bottom row by c, placing the result in the next column, and adding:

$$
c\;\begin{array}{|c|c|c|c|c|}
a_n & a_{n-1} & \dots & a_1 & a_0 \\
 & a_n\cdot c & \dots & a_2\cdot c+a_3\cdot c^2+\ldots+a_nc^{n-1} & a_1\cdot c+a_2\cdot c^2+\ldots+a_nc^n \\
\hline
a_n & a_{n-1}+\cdot a_n\cdot c & \dots & a_1+a_2\cdot c+a_3\cdot c^2+\ldots+a_nc^{n-1} & p(c)
\end{array}
$$

The vertical lines drawn above are not strictly necessary, but were included to make clear the division of columns as the table became full. To illustrate this method, consider the method for $s(1/2)$:

$$
1/2\;\begin{array}{|rrrrr}
1 & -6 & 10 & -6 & 4 \\
 & 1/2 & -11/4 & 29/8 & 19/16 \\
\hline
1 & -11/2 & 29/4 & -19/8 & 45/16
\end{array}
$$

To perform division by a monic binomial using this method, one follows essentially the same steps, using the root as c. For example, to begin factoring $4t^3-18t^2+20t-6$ from the observation that $t=1$ is a root, one perform the synthetic division:

$$
1\;\begin{array}{|rrrr}
4 & -18 & 20 & -6 \\
 & 4 & -14 & 6 \\
\hline
4 & -14 & 6 & 0
\end{array}
$$

The 0 confirms that 1 is a root, and also corresponds to the remainder when dividing $4t^3-18t^2+20t-6$ by $t-1$. The remaining numbers in the last row are the coefficients of the quotient, $4t^2-14t+6$. One then can factor this by usual methods for factoring a trinomial.