MATH 131, Fall 2019
Quiz 6 Alternate Solutions

1. A 100 milligram sample of an unidentified radioactive substance decays to 80 milligrams after 20 years. Find the half life of the substance, and re-express the mass in milligrams as a function of time in years since you obtained the original 100 milligrams in terms of the half-life.

Since the substance undergoes radioactive decay, we may assume its mass satisfies

$$m(t) = m_0 e^{kt} = 100e^{kt},$$

for some constant $k < 0$. Using the data about its decay over the first 20 years:

$$80 = 100e^{kt} \implies \frac{4}{5} = e^{k\cdot 20} \implies k = \frac{1}{20}\ln(4/5),$$

and so

$$m(t) = 100e^{(t/20)\ln(4/5)} = 100\left(\frac{4}{5}\right)^{t/20}.$$

We want to find λ such that $m(\lambda) = 50$ milligrams, and rewrite $m(t)$ in terms of λ. Then

$$\frac{1}{2} = \left(\frac{4}{5}\right)^{\lambda/20} \implies \lambda = 20\frac{\ln(1/2)}{\ln 4/5} = 20\frac{\ln 2}{\ln(5/4)} = 20\log_{5/4}(2).$$

Thus the half life is $20\log_{5/4} 2$ years. Note that $3 < \log_{5/4} 2 < 4$ since $125/64 < 2 < 525/256$, so the half life is more than 60 years but less than 80 years. Writing the mass in terms of the half life, one has

$$m(t) = 100\left(\frac{1}{2}\right)^{t/20\log_{5/4}(2)}.$$

2. A particle moves along the curve $xy = 1$ such that the particle approaches the y axis with a horizontal velocity component equal to the negative of its x coordinate. If the particle started at the point $(1, 1)$, find the time when the particle reaches a point P a distance of $1/3$ from the y axis, and find the velocity at which the particle's distance to the x axis increases as it passes through this point P.

The particle is moving along the hyperbola $xy = 1$. Since it's horizontal velocity, dx/dt is equal to the negative of it's position, we know $dx/dt = -x$ which implies $x = x_0e^{-t}$. Differentiating implicitly we can relate $\mathrm{d}x/\mathrm{d}t$ and $\mathrm{d}y/\mathrm{d}t$:

$$y\frac{\mathrm{d}x}{\mathrm{d}t} + x\frac{\mathrm{d}y}{\mathrm{d}t} = 0 \implies -xy + x\frac{\mathrm{d}y}{\mathrm{d}t} = 0 \implies \frac{\mathrm{d}y}{\mathrm{d}t} = y.$$

If the distance from the y axis is $1/3$, then the x coordinate is $x = 1/3$, and so the y coordinate must be the reciprocal, $y = 3$. Then the vertical velocity component is

$$\frac{\mathrm{d}y}{\mathrm{d}t} = 3\,.$$

To find the time to reach this point we use that $x = x_0 e^{-t}$ with starting position $(x_0, y_0) = (1, 1)$, and set $1/3 = x = e^{-t} \implies 3 = e^t \implies t = \ln 3$.