MATH 131, Fall 2019
Quiz 5 Solutions

1. Compute $\dfrac{\mathrm{d}}{\mathrm{d}x}\Big(\sin\big(2\cos^{-1}\sqrt{x}\,\big)\Big)$. You do not need to simplify.

To compute this derivative, we use the chain rule, as well as the fact that

$$\frac{\mathrm{d}}{\mathrm{d}x}\cos^{-1}u(x) = -\frac{u'(x)}{\sqrt{1-[u(x)]^2}}\,.$$

In case you forgot this, it can be derived from implicit differentiation.This derivation will be presented after the solution to this problem. Note that the innermost function in the composition $\sin\big(2\cos^{-1}\sqrt{x}\,\big)$ is $u=\sqrt{x}$, and the outermost function is $\sin w$ where $w = 2\cos^{-1}u$. By the chain rule

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}x}\Big(\sin\big(2\cos^{-1}\sqrt{x}\,\big)\Big) &= \cos\big(2\cos^{-1}\sqrt{x}\,\big)\cdot\frac{\mathrm{d}}{\mathrm{d}x}\Big(2\cos^{-1}\sqrt{x}\Big)\\ &= \cos\big(2\cos^{-1}\sqrt{x}\,\big)\cdot\frac{-2}{\sqrt{1-(\sqrt{x})^2}}\cdot\frac{\mathrm{d}}{\mathrm{d}x}\Big(\sqrt{x}\Big)\\ &= \cos\big(2\cos^{-1}\sqrt{x}\,\big)\cdot\frac{-2}{\sqrt{1-x}}\cdot\frac{1}{2\sqrt{x}}\\ &= -\frac{1}{\sqrt{x-x^2}}\cos\big(2\cos^{-1}\sqrt{x}\,\big)\,.\end{aligned}$$

We do not *need* to simplify further, but if we wished to, we could actually rewrite this as an algebraic function: using the cosine double angle formula $\cos 2\theta = 2\cos^2\theta - 1$, with $\theta = \cos^{-1}\sqrt{x}$, we have

$$\cos\big(2\cos^{-1}\sqrt{x}\,\big) = 2\cos^2\Big(\cos^{-1}\sqrt{x}\Big) - 1 = 2x-1\,,\;\; 0\le x\le 1\,.$$

The derivative then simplifies to

$$\frac{\mathrm{d}}{\mathrm{d}x}\Big(\sin\big(2\cos^{-1}\sqrt{x}\,\big)\Big) = -\frac{1}{\sqrt{x-x^2}}\cos\big(2\cos^{-1}\sqrt{x}\,\big) = \frac{1-2x}{\sqrt{x-x^2}}\,,\;\; 0<x<1\,.$$

Note that the derivative is undefined at $x=0$ and $x=1$, where there are vertical tangencies (as one can show by computing appropriate limits). One could also write the original function as an algebraic function:

$$\sin\big(2\cos^{-1}\sqrt{x}\,\big) = 2\sin\Big(\cos^{-1}\sqrt{x}\,\Big)\cos\Big(\cos^{-1}\sqrt{x}\,\Big) = 2\sqrt{1-x}\sqrt{x} = 2\sqrt{x-x^2}\,.$$

Using the power and chain rules the derivative of this matches that which we obtained above.
Now, the promised derivation of the derivative of $\cos^{-1}u(x)$. Write $y=\cos^{-1}u$, which, for u between -1 and 1 is equivalent to $u=\cos y$, with $0\le y\le\pi$. We will implicitly differentiate the latter equation, where both u and y are regarded as dependent on x.

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}x}\big(u\big) &= \frac{\mathrm{d}}{\mathrm{d}x}\big(\cos y\big)\\ u' &= -\sin(y)\cdot y'\\ y' &= -\frac{1}{\sin(y)} = -\frac{u'}{\sqrt{1-\cos^2(y)}} = -\frac{u'}{\sqrt{1-u^2}}\,,\end{aligned}$$

where we've used that $\sin^2(y)+\cos^2(y)=1$ to write our final expression in terms of u.

2. Find the equation of the line tangent to the curve $x^3 + 8xy^2 - y^5 = 1$ at $(1, 2)$. You may leave the equation in point-slope form if you wish.

Implicitly differentiating one obtains

$$3x^2 + 8y^2 + 16xyy' - 5y^4y' = 0$$

$$y' = \frac{3x^2 + 8y^2}{5y^4 - 16xy},$$

and the derivative at the point $(1, 2)$ is

$$y' = \frac{3 + 32}{80 - 32} = \frac{35}{48}.$$

Thus the tangent line equation is

$$y - 2 = \frac{35}{48}(x - 1).$$

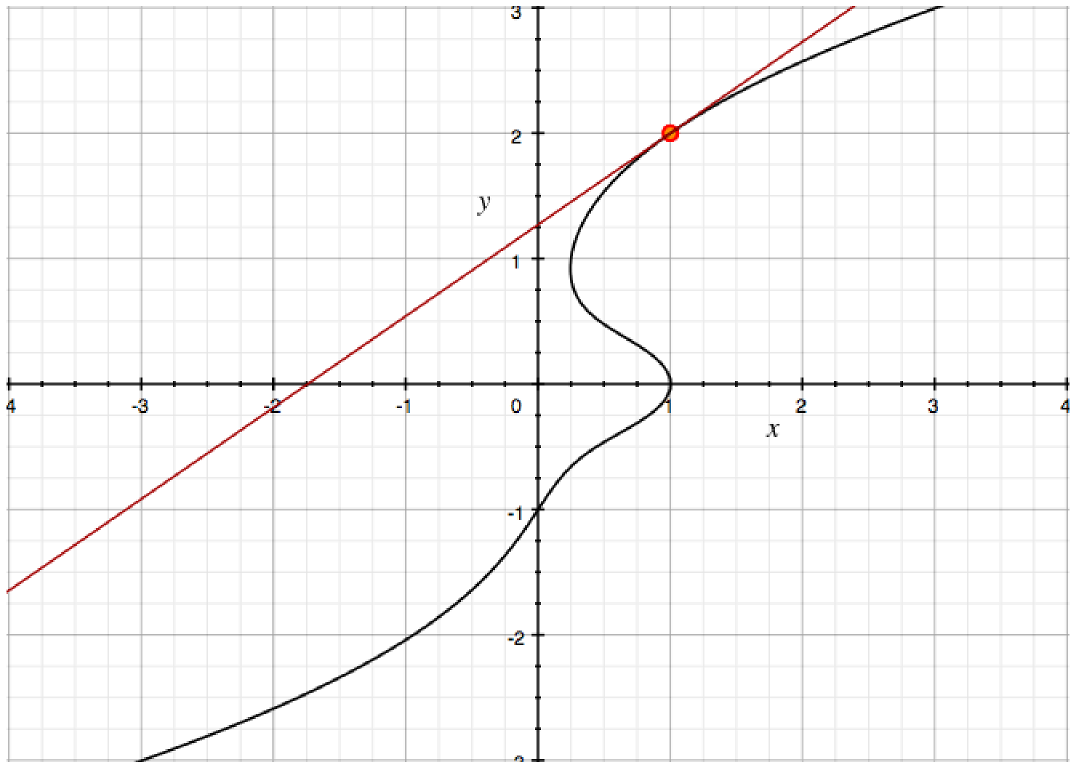

Figure 1: The curve $x^3 + 8xy^2 - y^5 = 1$ (black) and its tangent line at $(1, 2)$ (red).