MATH 131, Fall 2019
Quiz 5 Solutions

1. Compute $\frac{\mathrm{d}}{\mathrm{d}t}\Big(\cos\big(2\arctan t^3\big)\Big)$. You do not need to simplify.

To compute this derivative, we need the chain rule, together with the derivative formula for arctangent

$$\frac{\mathrm{d}}{\mathrm{d}t}\tan^{-1}(u(t)) = \frac{u'(t)}{1+[u(t)]^2}\,.$$

In case you forgot this, it can be derived from implicit differentiation. The derivation will be presented after the solution to this problem. Note that the innermost function is $u(t) = t^3$, and the outermost function is $\cos w$, with $w = 2\arctan(u)$. By the chain rule

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}\Big(\cos\big(2\arctan t^3\big)\Big) &= -\sin\big(2\arctan t^3\big)\cdot\frac{\mathrm{d}}{\mathrm{d}t}\big(2\arctan t^3\big)\\ &= -\sin\big(2\arctan t^3\big)\cdot\frac{2\cdot 3t^2}{1+(t^3)^2}\\ &= -\sin\big(2\arctan t^3\big)\frac{6t^2}{1+t^6}\,.\end{aligned}$$

We do not *need* to simplify further, but if we wished to, we could actually rewrite this as an algebraic function: using the sine double angle formula $\sin 2\theta = 2\sin\theta\cos\theta$, with $\theta = \arctan t^3$, we have

$$-\sin\big(2\arctan t^3\big)\frac{6t^2}{1+t^6} = -2\sin\big(\arctan t^3\big)\cos\big(\arctan t^3\big)\cdot\frac{6t^2}{1+t^6}\,.$$

But since $\theta = \arctan t^3$, we can write $t^3 = \tan\theta$. Taking t^3 to be the length of the side of a right triangle opposite an angle θ, and taking the length of the side adjacent to θ to have length 1, we discover that the hypotenuse should have length $\sqrt{1+t^6}$, and therefore

$$\sin\big(\arctan t^3\big) = \sin\theta = \frac{\text{opposite length}}{\text{hypotenuse length}} = \frac{t^3}{\sqrt{1+t^6}}\,,$$

$$\cos\big(\arctan t^3\big) = \cos\theta = \frac{\text{adjacent length}}{\text{hypotenuse length}} = \frac{1}{\sqrt{1+t^6}}\,,$$

and so the derivative then simplifies to

$$\frac{\mathrm{d}}{\mathrm{d}t}\Big(\cos\big(2\arctan t^3\big)\Big) = -2\sin\big(\arctan t^3\big)\cos\big(\arctan t^3\big)\cdot\frac{6t^2}{1+t^6} = \frac{-12t^5}{(1+t^6)^2}\,.$$

One could also write the original function as an algebraic function:

$$\cos\big(2\arctan t^3\big) = 2\cos^2(\arctan t^3) - 1 = \frac{2}{1+t^6} - 1\,.$$

Using the power and chain rules the derivative of this matches that which we obtained above.

Now, as promised, we'll compute the derivative $\arctan u(x)$ by implicit differentiation. Write $y = \arctan u$, which is equivalent to $u = \tan y$, with $-\pi/2 < y < \pi/2$. We will implicitly differentiate the latter equation, treating both u and y as dependent on x.

$$\begin{aligned} \frac{\mathrm{d}}{\mathrm{d}x}\big(u\big) &= \frac{\mathrm{d}}{\mathrm{d}x}\big(\tan y\big) \\ u' &= \sec^2(y) \cdot y' \\ y' &= \frac{u'}{\sec^2(y)} = \frac{u'}{1+\tan^2(y)} = \frac{u'}{1+u^2}\,, \end{aligned}$$

where we've used that $1 + \tan^2(y) = \sec^2(y)$ to write our final expression in terms of u.

2. Use logarithmic differentiation to find y' for $y = \sqrt[3]{x}^{\;2\arcsin(x)}$.

A caution: observe that the domain of this function is $[-1, 0) \cup (0, 1]$, but that upon taking natural logs, if one naively applies that $\ln x^2 = 2\ln x$, one loses the negative half of the domain. It's better to write $\ln x^2 = 2\ln|x|$. Taking the natural logarithm of y yields

$$\ln y = \ln\left[\sqrt[3]{x}^{\;2\arcsin(x)}\right] = \arcsin(x)\ln x^{2/3} = \frac{2}{3}\arcsin(x)\ln|x|\,.$$

Taking the derivative and using the product rule, we have

$$\frac{y'}{y} = \frac{2\ln|x|}{3\sqrt{1-x^2}} + \frac{2\arcsin(x)}{3x}\,.$$

Solving for y', using the original expression for y, yields

$$y' = y\left[\frac{2\ln|x|}{3\sqrt{1-x^2}} + \frac{2\arcsin(x)}{3x}\right] = x^{2\arcsin(x)/3}\left[\frac{2\ln|x|}{3\sqrt{1-x^2}} + \frac{2\arcsin(x)}{3x}\right]\,.$$