MATH 131, Fall 2019
Quiz 4 Solutions

1. Calculate the following derivatives by any means necessary. You do not need to simplify.

(a) $\dfrac{\mathrm{d}}{\mathrm{d}x}\Big((x-e^x)\tan(x)\Big)$,

(b) $f'(x)$ for $f(x) = \dfrac{x^3-1}{x^3+1}$,

(c) $g''(t)$ for $g(t) = \sin(t)\cos(t)$.

(a) Using the product rule:

$$\frac{\mathrm{d}}{\mathrm{d}x}\Big((x-e^x)\tan(x)\Big) = (1-e^x)\tan(x) + (x-e^x)\sec^2(x)\,.$$

(b) Using the quotient rule:

$$f'(x) = \frac{(x^3+1)(3x^2)-(x^3-1)(3x^2)}{(x^3+1)^2} = \frac{6x^2}{(x^3+1)^2}\,.$$

(c) There are a few approaches. Without chain rule, one can use repeated product rules:

$$g'(t) = (\sin t)'(\cos t) + (\sin t)(\cos t)' = \cos^2 t - \sin^2 t$$

$$\begin{aligned} g''(x) &= (\cos t)'(\cos t) + (\cos t)(\cos t)' - (\sin t)'(\sin t) - (\sin t)'(\sin t) \\ &= -4\sin t\cos t\,. \end{aligned}$$

If one knows the chain rule, the last calculation can be recovered using that $(\cos^2 t)' = 2\cos t(\cos t)' = -2\cos t\sin t$, and $(\sin^2 t)' = 2\sin t(\sin t)' = 2\cos t\sin t$.

Finally, if one recognizes the double angle identity $g(t) = \sin(t)\cos(t) = \frac{1}{2}\sin(2t)$, then one can use the chain rule twice:

$$g'(x) = \tfrac{1}{2}\Big(\sin(2t)\Big)' = \tfrac{1}{2}\cos(2t)\cdot(2t)' = \cos(2t)\,,$$

$$g''(x) = \Big(\cos(2t)\Big)' = -2\sin(2t)\,.$$

Note that the double angle formulae for sine and cosine show that these expressions are equal to those obtained by the repeated applications of the product rule.

2. Let $f(x) = \dfrac{x}{e^x} - 4\sin(x)$. Find $f^{(102)}(x)$, the 102nd derivative of $f(x)$.

Observe first that

$$\frac{\mathrm{d}}{\mathrm{d}x}\left(\frac{x}{e^x}\right) = \frac{1-x}{e^x}$$

either by the quotient rule or by applying the product and chain rules (since $x/e^x = xe^{-x}$). Calculating the next derivative we obtain

$$\frac{\mathrm{d}^2}{\mathrm{d}x^2}\left(\frac{x}{e^x}\right) = \frac{\mathrm{d}}{\mathrm{d}x}\left(\frac{1-x}{e^x}\right) = \frac{-e^x-(1-x)e^x}{e^{2x}} = \frac{-2+x}{e^x} = (x-2)e^{-x}\,.$$

The general pattern is

$$\begin{aligned}\frac{\mathrm{d}^n}{\mathrm{d}x^n}\left(\frac{x}{e^x}\right) &= \begin{cases}(n-x)e^{-x} & \text{if } n \text{ is odd,}\\ (x-n)e^{-x} & \text{if } n \text{ is even}\end{cases}\\ &= (-1)^n(x-n)e^{-x}\,.\end{aligned}$$

Next, using the periodicity of derivatives of $\sin x$, we deduce that

$$\frac{\mathrm{d}^{102}}{\mathrm{d}x^{102}}\sin x = \frac{\mathrm{d}^2}{\mathrm{d}x^2}\frac{\mathrm{d}^{100}}{\mathrm{d}x^{100}}\sin x = \frac{\mathrm{d}^2}{\mathrm{d}x^2}\sin x = -\sin x\,,$$

whence

$$f^{(102)}(x) = (x-102)e^{-x} + 4\sin x\,.$$