MATH 131, Fall 2019
Quiz 3 Solutions

1. Use *the limit definition* of the derivative to find $f'(x)$ for

$$f(x) = 2 + 4x - x^2 .$$

$$\begin{aligned} f'(x) &= \lim_{h\to 0} \frac{f(x+h)-f(x)}{h} = \lim_{h\to 0} \frac{2+4(x+h)-(x+h)^2-(2+4x-x^2)}{h} \\ &= \lim_{h\to 0} \frac{2+4x+4h-x^2-2xh-h^2-2-4x+x^2}{h} \\ &= \lim_{h\to 0} \frac{4h-2xh-h^2}{h} = \lim_{h\to 0} 4-2x-h \\ &= 4-2x . \end{aligned}$$

2. Find values of m and b such that the function

$$g(x) = \begin{cases} 2+4x-x^2 & \text{if } x \geq 0 \\ mx+b & \text{if } x < 0 \end{cases}$$

is differentiable when $x = 0$. You may use the results of the first question, but should still carefully justify differentiability of g at $x = 0$ by appealing to appropriate definitions or theorems.

I present two ways to think about the solution to this problem. The first is a more algorithmic approach that works whenever trying to arrange a piecewise function to be differentiable at transitions between pieces. The second method is less elaborate, or perhaps even obvious in hindsight, but doesn't aid when all the pieces are nonlinear.

Method 1: Recall, a function g is differentiable at $x = a$ if the derivative $g'(a)$ exists. Thus, we must guarantee that $g'(0)$ exists. Since g is defined as a piecewise function with two pieces meeting at $x = 0$, we must ensure that the limiting derivative values from the left and right agree:

$$g'(0) = \lim_{h\to 0} \frac{g(h)-g(0)}{h} = \lim_{h\to 0^+} \frac{g(h)-g(0)}{h} = \lim_{h\to 0^-} \frac{g(h)-g(0)}{h} .$$

For the right-sided limit, as we approach 0, the increment h is positive, and we use the quadratic branch of g, which is just the function $f(x)$ from problem 1:

$$g'(0) = \lim_{h\to 0^+} \frac{g(h)-g(0)}{h} = \lim_{h\to 0} \frac{f(h)-f(0)}{h} = f'(0) = 4-2(0) = 4 .$$

For the left-sided limit, as we approach 0, the increment is negative and we use the linear branch of g:

$$g'(0) = \lim_{h \to 0^-} \frac{g(h) - g(0)}{h} = \lim_{h \to 0^-} \frac{(mh + b) - 2}{h} = m + \lim_{h \to 0^-} \frac{b - 2}{h}\,.$$

First, note that for the limit to exist, we need $b = 2$, else the latter limit does not give a finite number. Another way to understand this is to recall that differentiability implies continuity, so in particular we must also guarantee that $\lim_{x \to 0} g(x) = g(0) = 2$. Thus, we must have $b = 2$ to ensure the correct limiting value of the linear form $mx + b$ as $x \to 0^-$. Then the limit for $g'(0)$ becomes

$$g'(0) = m + \lim_{h \to 0^-} \frac{2 - 2}{h} = m + 0 = m\,,$$

and we know that $g'(0)$ must equal 4, so $m = 4$, and

$$g(x) = \begin{cases} 2 + 4x - x^2 & \text{if } x \geq 0\,, \\ 4x + 2 & \end{cases}\,.$$

Method 2: As discussed above, g being differentiable at $x = 0$ means $g'(0)$ exists, and g must then necessarily be continuous at $x = 0$. Since g is continuous at $x = 0$ if and only if

$$\lim_{x \to 0^-} g(x) = g(0) = \lim_{x \to 0^+} g(x)\,,$$

we can deduce that the linear piece of g must pass through the point $\big(0, g(0)\big)$ to ensure continuity. To ensure differentiability, we need the slope of the line at the point where it contacts the parabola to match the derivative of the parabola there, which implies the line must be a tangent line! Less formally and more intuitively, if we imagine a particle traveling from the right along the parabolic part of the graph to the point $\big(0, g(0)\big)$, as the particle transitions to linear motion on left half of the graph it must be following the tangent direction in order for the motion to be smooth.

The above arguments then force us to conclude that $y = mx + b$ should have m and b chosen to ensure tangency to $y = f(x) = 2 + 4x - x^2$ at $(0, 2)$ whence $b = g(0) = 2$, and $m = f'(0) = 4$, which recovers the result obtained in method 1.