MATH 131, Fall 2019
Quiz 3 Solutions

his quiz has two sides!

1. Let c be a constant. Use *the limit definition* of the derivative to find $f'(x)$ for

$$f(x) = (cx)^2 - cx^3\,.$$

$$\begin{aligned}
f'(x) &= \lim_{h\to 0} \frac{f(x+h) - f(x)}{h} = \lim_{h\to 0} \frac{c^2(x+h)^2 - c(x+h)^3 - c^2x^2 + cx^3}{h} \\
&= \lim_{h\to 0} \left[c^2 \frac{(x+h)^2 - x^2}{h} - c\frac{(x+h)^3 - x^3}{h} \right] \\
&= c^2 \lim_{h\to 0} \left[\frac{x^2 + 2xh + h^2 - x^2}{h} \right] - c \lim_{h\to 0} \left[\frac{x^3 + 3x^2h + 3xh^2 + h^3 - x^3}{h} \right] \\
&= c^2 \lim_{h\to 0} \big(2x + h\big) - c \lim_{h\to 0} \big(3x^2 + 3xh + h^2\big) \\
&= c^2(2x) - c(3x^2) \\
&= 2c^2x - 3cx^2 = (cx)(2c - 3x)\,.
\end{aligned}$$

2. Find all values of c such that the function

$$g(x) = \begin{cases} (cx)^2 - cx^3 & \text{if } x \le 1 \\ 2x & \text{if } x > 1 \end{cases}$$

is differentiable when $x = 1$. You may use the results of the first question, but should still carefully justify differentiability of g at $x = 1$ by appealing to appropriate definitions or theorems.

Recall, a function g is differentiable at $x = a$ if the derivative $g'(a)$ exists. Thus, we must guarantee that $g'(1)$ exists. Since g is defined as a piecewise function with two pieces meeting at $x = 1$, we must ensure that the limiting derivative values from the left and right agree:

$$g'(1) = \lim_{x\to 1} \frac{g(x) - g(1)}{x - 1} = \lim_{x\to 1^-} \frac{g(x) - g(1)}{x - 1} = \lim_{x\to 1^+} \frac{g(x) - g(1)}{x - 1}\,.$$

For the right-sided limit, as we approach 1, we use the linear piece of g:

$$g'(1) = \lim_{x\to 1^-} \frac{g(x) - g(1)}{x - 1} = \lim_{x\to 1^-} \frac{2x - g(1)}{x - 1}\,.$$

The limit will exist and return a real number if and only if

$$g(1) = \lim_{x\to 1^+} g(x) = \lim_{x\to 1^+} 2x = 2\,,$$

in which case the limit for $g'(1)$ is

$$\lim_{x\to 1^+} \frac{2x-1}{x-1} = 2 \lim_{x\to 1^+} \frac{x-1}{x-1} = 2\,.$$

For the left-sided limit, as we approach 1, we use the cubic branch of g, which is the function $f(x) = (cx)^2 - cx^3$ from the first question:

$$g'(1) = \lim_{x\to 1^-} \frac{g(x)-g(1)}{x-1} = \lim_{h\to 0^-} \frac{g(1+h)-g(1)}{h} = \lim_{h\to 0^-} \frac{f(1+h)-f(1)}{h} = f'(1)\,,$$

In particular, this limit is just the derivative calculated in the first part, evaluated when $x = 1$:

$$f'(1) = 2c^2 - 3c\,.$$

Since the left- and right-sided limits must agree for g to be differentiable at $x = 1$, we require that

$$2c^2 - 3c = 2 \text{ or equivalently, } 2c^2 - 3c - 2 = 0\,.$$

Factoring yields $2c^2 - 3c - 2 = (2c+1)(c-2)$ which gives two potential c values: $c = 2$ or $c = -1/2$. However, we know $g(1)$ must equal 2, and if we choose $c = -1/2$, we get

$$g(1) = \frac{1}{4} + \frac{1}{2} = \frac{3}{4} \neq 2\,,$$

in contradiction with our conclusions above. Thus the only solution is $c = 2$, which gives

$$\begin{cases} 4x^2 - 2x^3 & \text{if } x \le 1 \\ 2x & \text{if } x > 1 \end{cases}.$$

Another way to arrive at this final conclusion is to recall that differentiability implies continuity, and so for g to be differentiable, we needed continuity (which is imposed when we realize that the right-sided derivative limit requires a value of $g(1)$ consistent with the limit of the linear right piece of the graph). From this perspective, we found the *only* value of c for which a cubic polynomial of the form $y = (cx)^2 - cx^3$ has tangent line $y = 2x$, tangent when $x = 1$ (so y necessarily has to equal 2 at the point of tangency).