MATH 131, Fall 2019
Quiz 2 Solutions

1. Use continuity and the limit laws to evaluate the limit $\lim_{x\to 0} \cos\left(\frac{\pi}{2x} - \frac{\pi}{x(x+2)}\right)$. You must show all steps for full credit.

Since $\cos(x)$ is continuous throughout its domain, we can move the limit operator inside the cosine. We then must work to simplify the rational function inside the cosine. Note that finding the common denominator allows us to use replacement to evaluate the limit:

$$\begin{aligned}
\lim_{x\to 0} \cos\left(\frac{\pi}{2x} - \frac{\pi}{x(x+2)}\right) &= \cos\left(\lim_{x\to 0}\left[\frac{\pi}{2x} - \frac{\pi}{x(x+2)}\right]\right) \\
&= \cos\left(\lim_{x\to 0}\left[\frac{\pi(x+2)}{2x(x+2)} - \frac{2\pi}{x(x+2)}\right]\right) \\
&= \cos\left(\lim_{x\to 0}\frac{\pi x + 2\pi - 2\pi}{2x(x+2)}\right) \\
&= \cos\left(\lim_{x\to 0}\frac{\pi}{2(x+2)}\right) \\
&= \cos\left(\frac{\pi}{2(0+2)}\right) \\
&= \cos\left(\frac{\pi}{4}\right) = \frac{\sqrt{2}}{2}
\end{aligned}$$

2. Show that the polynomial $p(x) = x^5 + x^3 - 2x + 1$ has a real root. (You do not need to find the root, but must carefully justify its existence.)

It will suffice to apply IVT to conclude that the function takes the value 0 for some number r which falls between x values that produce opposing signs in the value of p.

Observe that $p(0) = 1 > 0$. Thus, we next seek an x which makes $p(x) < 0$. Any sufficiently negative x will work, such as $x = -2$, which yields

$$p(-2) = (-2)^5 + (-2)^3 - 2(-2) + 1 = -32 - 8 + 4 + 1 = -35.$$

Since $p(x)$ is a polynomial and thus continuous on $\mathbb{R}$, it is certainly continuous on the interval $[-2, 0]$. By the intermediate value theorem, over the open interval $(-2, 0)$ the continuous function $p(x)$ must take on every value N with $-35 < N < 1$. In particular, since $p(-2) = -35 < 0 < 1 = p(0)$, we can conclude that $p(r) = 0$ for some real root r in the interval $(-2, 0)$.