MATH 131, Fall 2019
Quiz 1 Solutions

1. Evaluate the limit

$$\lim_{x\to 0}\left(\frac{x}{\sin x}-1\right)\cos\left(\frac{1}{x}\right).$$

You must use the limit laws to justify your limit calculation. (You can invoke the value of $\lim_{x\to 0}\sin(x)/x$ without proof.)

We will make use of the fact that $\lim\limits_{x\to 0}\dfrac{\sin x}{x}=1$, and apply limit laws to show that

$$\lim_{x\to 0}\left(\frac{x}{\sin x}-1\right)=0.$$

Indeed:

$$\begin{aligned}\lim_{x\to 0}\left(\frac{x}{\sin x}-1\right)&=\lim_{x\to 0}\frac{x}{\sin x}-\lim_{x\to 0}1\\&=\frac{1}{\lim_{x\to 0}\sin(x)/x}-1\\&=1-1=0\,.\end{aligned}$$

Next, we observe that, although $\lim_{x\to 0}\cos(1/x)$ does not exist, we can still apply the squeeze theorem and compute the limit of the product. Recall cosine is bounded:

$$-1\le\cos(1/x)\le 1$$

for every nonzero real number x. Now, since $\frac{x}{\sin x}\ge 1$ for sufficiently small x,

$$1-\frac{x}{\sin x}\le\left(\frac{x}{\sin x}-1\right)\cos\left(\frac{1}{x}\right)\le\frac{x}{\sin x}-1$$

for x sufficiently close to 0. Thus, since $1-\frac{x}{\sin x}$ and $\frac{x}{\sin x}-1$ both approach 0 as x approaches 0, by the squeeze theorem

$$\lim_{x\to 0}\left(\frac{x}{\sin x}-1\right)\cos\left(\frac{1}{x}\right)=0\,.$$