Fundamental groups of number fields

Farshid Hajir

In this mostly expository lecture aimed at low-dimensional topologists, I outlined some basic facts and problems of algebraic number theory. My focus was on one particular aspect of the rich set of analogies between number fields and 3-manifolds dubbed *Arithmetic Topology.* Namely, I discussed the role played in number theory by "fundamental groups" of number fields, and related some of the history of the subject over the past fifty years, since the unexpected discovery by Golod and Shafarevich of number fields with infinite fundamental group; see the monograph of Neukirch, Schmidt, Wingberg [10] for a comprehensive account. A conjecture of Fontaine and Mazur [3] has been influential in stimulating work on the structure of these infinite fundamental groups in recent years. I presented a formulation of this conjecture as it relates to the asymptotic growth of discriminants [6]. This discussion then served as motivation for a question about non-compact, finite-volume, 3-manifolds inspired by the following dictionary.

Topology	Arithmetic
M non-compact, finite-volume hyperbolic 3-manifold	K a number field or, more precisely, $X = \mathrm{Spec}\mathcal{O}_K$
universal cover $\widetilde{M}$	$\widetilde{K}$ = max. unramified extension of K
fundamental group $\pi_1(M)$	$\mathrm{Gal}(\widetilde{K}/K) \approx \pi_1^{\mathrm{et}}(X)$
Klein-bottle cusps of M	Real ("unoriented") places of K
Torus cusps of M	Complex ("oriented") places of K
$r_1 = \#$ Klein-bottle cusps of M	$r_1 = \#$ Real places of K
$r_2 = \#$ Torus cusps of M	$r_2 = \#$ Complex places of K
$r = r_1 + r_2 = \#$ cusps of M	$r = r_1 + r_2 = \#$ places of K at ∞
$n = r_1 + 2r_2$ = weighted $\#$ cusps	$n = r_1 + 2r_2 = [K : \mathbb{Q}]$
$\mathrm{vol}(M)$ = volume of M	$\log \lvert d_K \rvert$, d_K = discriminant of K

There are multiple accounts of the dictionary of arithmetic topology; these include Reznikov [12], Ramachandran [11], Deninger [2], Morin [9], and Morishita [8]. For the subtle distinction between $\mathrm{Gal}(\widetilde{K}/K)$ and the étale fundamental group of Spec $\mathcal{O}_K$ when K is not "orientable," i.e. $r_1(K) \neq 0$, see Ramachandran [11]. I hit upon the analogy between cusps and infinite places as well as between volumes and discriminants during several conversations with Champanekar and Dunfield at the 2010 Oberwolfach meeting on *Low-dimensional topology and number theory*, and wish to thank them both for their patient explanations to a non-specialist. For Ramachandran's justification of the cusps-places analogy, see section 2 of Deninger [2]. As justification for drawing a parallel between volumes for hyperbolic 3-manifolds (or more generally Gromov norms of 3-manifolds) with logarithmic discriminants for number fields, I limit myself here to appealing to the "Riemann-Hurwitz genus formula for number fields,"

$$\log \lvert d_L \rvert = [L : K] \log \lvert d_K \rvert + \log \lvert \mathbb{N}_{K/\mathbb{Q}} d_{L/K} \rvert$$

where $d_{L/K}$ is the relative discriminant of L/K. Thus, when L/K is a covering, i.e. is unramified, the volume scales up by a factor of $[L : K]$, just as with coverings of manifolds. The relative discrimiant $d_{L/K}$ is made up of a "wild" component corresponding to prime ideals of K that divide a prime divisor of $[L : K]$ and a "tame" component. While the latter is easy to compute, the former can be quite intricate.

The Riemann-Hurwitz formula relates the existence of coverings to the rate of growth of discriminants. It was this fact which led Minkowski to create his "geometry of numbers" for the purpose of proving the following conjecture of Kronecker: $\widetilde{\mathbb{Q}} = \mathbb{Q}$. Minkowski actually showed much more, namely that the discriminant grows exponentially with the degree. For this reason, we define a normalized discriminant for number fields $\nu(K) := \frac{\log |d_K|}{[K:\mathbb{Q}]}$, called the logarithmic root discriminant. This quantity remains constant in unramified extensions and remains bounded for extensions which are *tamely* ramified at a finite number of primes.

In his proof that discriminants grow exponentially with the degree, Minkowski found that real and complex places give different contributions. Namely, he found constants $A > B > 0$ such that $\log |d_K| \geq Ar_1 + Br_2 - \delta(n)$, where $\delta(n)$ is a small error term that is in $o(n)$ as $n = r_1 + 2r_2 \to \infty$. To reformulate this type of bound in the language of normalized discriminants, we introduce the parameter $t = r_1/n$. The best known values of A, B come from the study of Dedekind zeta functions of number fields. If we admit the Generalized Riemann Hypothesis for these zeta functions, we have

$$\nu(K) \geq \log(8\pi) + \gamma + t\pi/2 - \varepsilon(n) \tag{1}$$

with an explicit error term $\varepsilon(n)$ that tends to 0 with $n = [K : \mathbb{Q}]$.

If we follow the analogy introduced in the table above, we are led to the question: does the volume of an r-cusped hyperbolic 3-manifold grow linearly with r? The answer is yes. Indeed, we have the following theorem of Adams [1]: *If M is an r-cusped hyperbolic 3-manifold, then* $\mathrm{vol}(M) \geq v_3 r$ *where v_3 is the volume of the regular ideal tetrahedron.*

We note that Adams' proof relies on Minkowski's geometry of numbers. Even without this fact as a provocation, it is natural for a number-theorist to wonder whether Adams' theorem can similarly be refined for contributions from torus cusps and Klein-bottle cusps. A somewhat vague form of the question is: What are the optimal values of positive constants v_1 and v_2 such that every hyperbolic 3-manifold having r_1 Klein-bottle and r_2 torus cusps satisfies $\mathrm{vol}(M) \geq r_1 v_1 + r_2 v_2$? To make the question more precise, let us define, for an r-cusped 3-manifold M with r_1 Klein bottle cusps and $r - r_1 = r_2$ torus cusps, the *orientation type* t of M to be $t = r_1/r$ and its *normalized volume* to be $\nu(M) := \mathrm{vol}(M)/r$. It is clear that we intend $\nu(M)$ to be a reasonable analogue of the logarithmic root discriminant for number fields.

In number theory, the estimate (1) is of great importance; in particular, an interesting problem to determine whether the constants in the linear function bounding the normalized discriminant from below are optimal; this is measured by a function defined by Martinet (see [7] and also [5]). As an analogue of the Martinet function, we define a function $\mathscr{A}(t)$ as follows: For a rational number $t \in [0,1]$, define

$$\mathscr{A}(t) = \inf_{M \text{ of type } t} \nu(M),$$

the infimum being taken over all hyperbolic 3-manifolds of orientation type t.

The question then is to determine (upper and lower bounds for) $\mathscr{A}(t)$. If for no other reason than for the analogy with asymptotic problems of this type in number theory and many other contexts (graph theory, coding theory, curves over finite fields etc., see [4]), it would be very interesting if it can be established that $\mathscr{A}(t)$ is a linear function, or that it meets a fixed linear lower bound for many values of t.

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