SS 1.5. (a) Suppose \( \Omega = \Omega_1 \coprod \Omega_2 \) with non-empty open subsets \( \Omega_i \).

Let

\[
S = \{ t \in [0,1] : 0 \leq s < t \text{ implies } z(s) \in \Omega_1 \},
\]

and \( t^* = \sup S \). Note that \( t^* \neq 0,1 \) by the non-emptiness assumption on the \( \Omega_i \).

**Claim 1.** For \( 0 \leq t_0 < t^* \), we have \( z(t_0) \in \Omega_1 \). Indeed, by definition of \( t^* \), \( t_0 \) is not an upper bound for \( S \), hence \( t_0 < t \) for some \( t \in S \). By the defining property of \( S \), we then have \( z(t_0) \in \Omega_1 \).

**Claim 2.** \( z(t^*) \notin \Omega_2 \). Otherwise, since \( \Omega_2 \) is open, we could find \( r > 0 \) with \( D_r(z(t^*)) \subset \Omega_2 \). Let \( X = z^{-1}(D_r(z(t^*))) \) be the inverse image of this disc; by the continuity of \( z \), \( X \) is an open subset of \([0,1]\) and it contains \( t^* \) of course, so for some \( \delta > 0 \), we have \( t^* - \delta < s < t^* + \delta \) implies \( z(s) \in \Omega_2 \). This is in violation of Claim 1 (e.g. take \( t_0 = t^* - \delta/2 \)), so we have established Claim 2. [Note: it turned out to be important to know that \( t^* \neq 0 \! \! \! 1 \)! [Note on previous note: “0!” above means “0 factorial,” which would be 1.]]

**Claim 3.** \( z(t^*) \notin \Omega_1 \). Suppose otherwise. As above, we choose \( u > 0 \) such that \( D_u(z(t^*)) \subset \Omega_1 \), with inverse image \( Y = z^{-1}(D_u(z(t^*))) \) under \( z \), an open non-empty subset of \([0,1]\). We take \( \epsilon > 0 \) such that the \( \epsilon \)-neighborhood of \( t^* \) is entirely contained in \( Y \) (Note: now we are using the fact that \( t^* \neq 1 \! \! \! 1 \))(Note on previous note: here reading “1!” either way is acceptable). Thus, \( (t^* - \epsilon, t^* + \epsilon) \subset Y \). So, \( T = t^* + \epsilon \) has the property that for all \( s \in [0,T) \), \( z(s) \in \Omega_1 \). In other words, \( T \in S \) and \( T > t^* = \sup S \). This is a contradiction.

Claims 2 and 3 are not compatible with the partition \( \Omega = \Omega_1 \coprod \Omega_2 \), hence \( \Omega \) must be connected.

(b) For \( z, w \in \Omega \), we’ll say \( z \) is \( \Omega \)-connectable to \( w \) if there is a smooth path from \( z \) to \( w \) which is entirely contained in \( \Omega \).

We will use the fact that if \( w \in D_r(z_0) \), then the straight line path connecting \( z \) to \( w \) lies entirely in \( D_r(z_0) \). Fix \( w \in \Omega \). Let \( \Omega \) consist of the points in \( \Omega \) which are “\( \Omega \)-connectable” to \( w \) and \( \Omega \) is the set of those points that are not. Clearly, these \( \Omega \) partition \( \Omega \). The constant path \( z : [0,1] \to \{w\} \) shows \( w \) is \( \Omega \)-connected to itself, so \( w \in \Omega_1 \) and \( \Omega_1 \) is non-empty. Now, let’s show it is open. Suppose \( z_0 \in \Omega_1 \). Take \( \delta > 0 \) small enough so that \( D_r(z_0) \subset \Omega \). For all \( z \in D_r(z_0) \), \( z \) is \( D_r(z_0) \)-connectable to \( z_0 \), hence \( \Omega \)-connectable to \( z_0 \). Concatenating this path to one from \( z_0 \) to \( w \), we see that \( z \in \Omega_1 \). Thus, \( \Omega_1 \) is open. Now, suppose there exists \( z_0 \in \Omega_2 \). Take \( \delta > 0 \) small so \( D_{\delta}(z_0) \subset \Omega \). For \( z \) in this disc, should \( z \) be \( \Omega \)-connectable to \( w \), then \( z_0 \) would share the same fate (by concatenation with the straight-line path between \( z_0 \) and \( z \)), contradicting \( z_0 \in \Omega_2 \). Thus \( D_\delta(z_0) \subset \Omega_2 \), hence \( \Omega_2 \) is open as well. But \( \Omega \) is connected, and \( \Omega_1 \) is non-empty, so \( \Omega_2 \) must be empty. In other words, \( \Omega \) is path-connected.
SS 1.7. (a) Let \( f(w, z) = (w - z)/(1 - wz) \). Suppose \(|\zeta| = 1\). Then \( f(\zeta w, \zeta z) = \zeta f(w, z) \). You could say, \( f \) is “\( S^1 \)-homogeneous of degree 1.” For a given pair \((w, z)\) in the closure of the unit disc, if we choose \( \zeta = e^{-i\theta} \) where \( \theta = \arg z \), we then have \((\zeta w, \zeta z)\) is just the rotation of the pair that takes \( z \) to the real line; it also rotates the value of \( f \) by the same angle, which of course does not change its modulus. In other words, \(|f(w, z)| = |f(\zeta w, \zeta z)|\) and \( \zeta z \in \mathbb{R} \), so we’ve reduced to the case where \( z = r \in \mathbb{R} \). In this case, \(|f(z, w)| \leq 1\) if and only if \((w - r)(\overline{w} - r) \leq (1 - wr)(1 - wr)\). A little algebra now takes care of the rest of (a).

Another easy alternate proof can be given later in the term using the Maximum modulus principle.

Suppose \( f = u + iv \) is holomorphic on an open set \( \Omega \) and has constant modulus there, say \( C \). If \( C = 0 \), then \( f \) is identically 0, hence constant, so let’s assume \( C \neq 0 \). We have \( u^2 + v^2 = C \neq 0 \). Differentiating implicitly, we have \( uu_x + vv_x = 0 = uu_y + vv_y \). Suppose \( f'(z_0) = A \neq 0 \) for some \( z_0 \in \Omega \). Then the Jacobian matrix has non-zero determinant \( A \). Multiplying the above system by its inverse, we find \( u, v \) both vanish at \( z_0 \), so \( f \) vanishes at \( z_0 \) as well, but \( f \) has constant, non-zero, modulus, and this is a contradiction. Thus \( f' \) vanished on \( \Omega \) and we can apply corollary ?? to conclude that \( f \) is constant.