Here is some clarification for the proof of Theorem 5.3 from Chapter 2 of Stein–Shekarchi.

**Theorem 0.1.** If \( f_1, f_2, f_3, \ldots \) is a sequence of functions holomorphic on an open set \( \Omega \subseteq \mathbb{C} \) which converge to a function \( f \) on \( \Omega \) and if this convergence is uniform on all compact subsets of \( \Omega \), then

1. the function \( f \) is holomorphic on \( \Omega \), and
2. the sequence of derivatives \( f'_n \) converges uniformly to \( f' \) on all compact subsets of \( \Omega \).

**Proof of (2).** This theorem was apparently first given by Weierstrass, not Hurwitz, as I mistakenly said in class. The proof of (1) using Morera from the book or lecture is straightforward. Here is a fussy elaboration of the proof given in Stein–Shekarchi. Let \( \Gamma \) be a compact subset of \( \Omega \). Since \( \Gamma \) is compact and \( \Omega \) is open, there exists a chain of sets\( \Gamma \subseteq \Omega' \subseteq \Gamma' \subseteq \Omega \), with \( \Omega' \) open and \( \Gamma' \) compact. (Easy verification left to the reader). Now there exists \( \delta > 0 \) such that \( \Gamma' \subset \Omega'_\delta \), where \( \Omega'_\delta := \{ z \in \Omega' \mid D_\delta(z) \subset \Omega' \} \).

We will now show that \( f'_n \to f' \) uniformly on \( \Omega'_\delta \), which is all we need do since \( \Gamma \subset \Omega'_\delta \).

**Claim.** If \( F(z) \) is holomorphic on \( \Omega \) (or even just on \( \Omega' \)), then

\[
\sup_{z \in \Omega'_\delta} |F(z)| \leq \frac{1}{\delta} \sup_{\zeta \in \Omega'} |F(\zeta)|.
\]

Note that although \( \Omega'_\delta \) and \( \Omega' \) are not compact, they are both contained in the compact set \( \Gamma' \), hence the two sup’s above are well-defined real numbers.

To prove the claim, we apply Cauchy’s formula just as in the book and lecture: For all \( z \in \Omega'_\delta \), \( D_\delta(z) \subset \Omega' \), so \( C_\delta(z) \subset \Omega' \), giving us

\[
|F'(z)| = \left| \frac{1}{2\pi i} \int_{C_\delta(z)} \frac{F(\zeta)}{\zeta - z} d\zeta \right| \\
\leq \frac{1}{2\pi} \sup_{z \in \Omega'} |F(z)| \frac{2\pi\delta}{\delta^2} \\
\leq \frac{1}{\delta} \sup_{z \in \Omega'} |F(z)|,
\]

verifying the claim.
Now suppose $\epsilon > 0$ is given. Since $f_n \to f$ uniformly on $\Gamma'$ there exists an integer $N$ such that

$$|f_n(z) - f(z)| < \delta/2$$

for all $z \in \Gamma'$ and all $n \geq N$.

Then, for $n \geq N$ and $z \in \Omega_\delta$, we apply the claim to $F(z) := f_n(z) - f(z)$ and find that

$$|F'(z)| = |f'_n(z) - f'(z)| \leq \frac{1}{\delta} \sup_{z \in \Omega_\delta} |f_n(z) - f(z)| \leq \frac{1}{\delta} \sup_{z \in \Gamma'} |f_n(z) - f(z)| \leq \frac{1}{\delta} \frac{\delta \epsilon}{2} < \epsilon.$$

We have shown that $f'_n \to f'$ uniformly on $\Omega_\delta'$.  

The most common application of the above theorem is contained in:

**Corollary 0.2.** If the functions $e_0(z), e_1(z), e_2(z), \ldots$ are holomorphic on the open set $\Omega$ and the series $\sum_{n \geq 0} e_n(z)$ converges to a function $f$ on $\Omega$ and it does so uniformly on compact subsets of $\Omega$, then $f$ is holomorphic on $\Omega$ and $f'(z) = \sum_{n \geq 0} e'_n(z)$ for $z \in \Omega$.

Later, after we prove the Maximum Modulus and Argument Principles, we’ll be able to obtain the following theorems.

**Theorem 0.3.** If $e_n(z)$ are holomorphic on $|z| < R$ and $f(z) = \sum_{n \geq 0} e_n(z)$ converges uniformly on the circles $C_r(0)$ for all $0 < r < R$, then $f$ is holomorphic on $|z| < R$.

The proof follows from the Corollary once we invoke the maximum modulus principle.

**Theorem 0.4** (Hurwitz). If $f_n$ are holomorphic and non-vanishing on $\Omega$ and converge uniformly to $f$ on compact subsets of $\Omega$, then $f$ is either identically zero or non-vanishing on $\Omega$.

We’ll be able to deduce this Hurwitz from Weierstrass after we prove the Argument Principle.

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Alternative Hint for Exercise 2 from Chapter 2 of Stein-Shekarchi.

To evaluate $I = \int_0^\infty \frac{\sin(x)}{x} dx$, we can note that

$$I = \frac{1}{2i} \lim_{\epsilon \to 0} \left( \int_{-\infty}^{-\epsilon} \frac{e^{ix}}{x} dx + \int_{\epsilon}^{\infty} \frac{e^{ix}}{x} dx \right).$$

Now we integrate $e^{iz}/z$ over the indented semi-circle, consisting of a radius $R$ semi-circle in the upper half plane and the real axis between $-R$ and $R$ with the exception of a radius $\epsilon$ circular “bump” around the origin. The integral over the radius $R$ semi-circle goes to 0 as $R$ goes to 0, as you can bound its modulus from above by

$$\int_0^\pi e^{-R \sin(\theta)} d\theta = 2 \int_0^{\pi/2} e^{-R \sin \theta} d\theta.$$

Note that we are using the uniform convergence of $f_n$ on the larger set $\Gamma$ to deduce the uniform convergence of $f_n'$ on the proper subset $\Gamma$.
For $0 \leq \theta \leq \pi/2$, there exists a constant $C > 0$ such that $\sin(\theta) \geq C\theta$, so that the integral above is bounded above by

$$2 \int_0^{\pi/2} e^{-RC\theta} \, d\theta,$$

which you can evaluate explicitly. For the integral on the circle of radius $\epsilon$, $e^{iz}/z = 1/z + G(z)$ where $G(z)$ is holomorphic at 0. Since $G(z)$ is bounded by a constant, say $B$, near $z = 0$, we get for all sufficiently small $\epsilon > 0$,

$$\left| \int_{C_\epsilon} e^{iz}/z \, dz - \int_{C_\epsilon} dz/z \right| < B\pi\epsilon$$

so upon letting $\epsilon \to 0$, we get what we want.