1. If $G$ is a group and $H$ is a subgroup of it, then a right coset of $H$ in $G$ is a subset of the form $Hg = \{hg | h \in H\}$ for some $g \in G$. In other words, this is just like left cosets but the element is multiplied on the right. We denote by $H \setminus G$ the set of right cosets of $H$ (recall that $G/H$ is the set of left cosets of $H$). Give an example of a group $G$ and a subgroup $H$ and an element $g \in G$ such that $Hg \neq gH$. Hint: don’t pick a commutative group!!

2. In the following examples, for each group $G$, give a list of the different left cosets of $H$ as a subset of $G$, then a list of the different right cosets of $H$. For example, if $G = \mathbb{Z}$ and $H = 2\mathbb{Z}$, then $G/H = \{\{2k | k \in \mathbb{Z}\}, \{2k + 1 | k \in \mathbb{Z}\}\}$ has two elements, the set of even numbers and the set of odd numbers. The left and right cosets coincide so $H \setminus G$ is the same as $G/H$.
   a. $G = \mathbb{Z}$, $H = 6\mathbb{Z}$.
   b. $G = S_3$, $H = \{e, \sigma, \sigma^2\}$ where $\sigma = (123)$ is the 3-cycle sending 1 to 2, 2 to 3, 3 to 1.
   c. $G = S_3$, $H$ is the set of permutations that fix the number 2.
   d. $G = GL_2(\mathbb{Z})$; this is the group of $2 \times 2$ matrices with integer entries which have determinant either $+1$ or $-1$. Verify that $G$ is a group and that $H = SL_2(\mathbb{Z})$ is a subgroup of $G$. Now list $G/H$.

3. **Conjugation.** If $G$ is a group, and $g \in G$, then we define a map $c_g : G \to G$ by $c_g(x) = gxg^{-1}$. Show that $c_g$ is always an isomorphism. What is the relationship between $c_g$ and $c_{g^{-1}}$? Is it correct to say that the kernel of $c_g$ is the set of elements of $G$ that commute with $g$?

4. Suppose $G$ is a group, and $H, K$ are two subgroups of $G$. Show that their intersection $H \cap K$ is also a subgroup of $G$. Show that if $H$ and $K$ are normal subgroups of $G$, then $H \cap K$ is a normal subgroup of $G$.

5. A subgroup $H$ of a group $G$ is called a normal subgroup of $G$ if for all $g \in G$ and all $h \in H$, $ghg^{-1} \in H$. Show that $H$ is a normal subgroup of $G$ if and only if every right coset of $H$ coincides with the corresponding left coset of $H$, i.e. for all $g \in G$, $gH = Hg$.

6. Suppose $f : G_1 \to G_2$ is a group homomorphism. Prove that ker$(f)$ is a normal subgroup of $G_1$. [We already know it’s a subgroup; now show that it’s normal].

7. Suppose $f : G_1 \to G_2$ is a group homomorphism. Suppose $x \in G_1$ has order $k$ and $f(x)$ has order $\ell$ in $G_2$. Show that $\ell | k$, i.e. $k$ is a multiple of $\ell$. 


8. Suppose \( f : G_1 \rightarrow G_2 \) is a group homomorphism, where \( G_1 \) and \( G_2 \) are two finite groups satisfying \( gcd(|G_1|, |G_2|) = 1 \). Show that \( f(x) = e_2 \) for all \( x \in G_1 \), i.e. \( f \) is the trivial homomorphism. Hint: use the previous problem.

HERE’S AN EXTRA PROBLEM FOR THOSE WHO WANT TO DO IT BUT IT’S NOT PART OF THE REGULAR ASSIGNMENT: Suppose \( H, K \) are subgroups of \( G \) and that \( H \subseteq K \). Show that \( H \) is a subgroup of \( K \). Is it true that if \( K \) is a normal subgroup of \( G \) and \( H \) is a normal subgroup of \( K \), then \( H \) is a normal subgroup of \( G \)? Either give a proof, or find a counterexample. Hint: take \( G = S_3 \) and list all its subgroups. Now play with different \( H \subset K \subset G \) to get a feeling for the problem.