1. Define the following terms completely and accurately.
A map \( f : G \to K \) is a homomorphism if ...
If \( f : G \to K \) is a homomorphism, then \( \ker(f) \) is ...
If \( G \) is a group and \( H \) is a subgroup of \( G \), then a left coset of \( H \) in \( G \) is ...
If \( G \) is a group and \( H \) is a subgroup of \( G \), then \( G/H \) is ...
If \( G \) is a group and \( H \) is a subgroup of \( G \), then the index of \( H \) in \( G \) (denoted by \([G : H]\)) is ...
A subgroup \( H \) of \( G \) is called normal if ...
The “natural map” from \( G \) to \( G/H \) is defined by ...
If \( H \) is a normal subgroup of \( G \), we give \( G/H \) a natural group structure by defining \( aH \ast bH = ... \)
The first isomorphism theorem states that ....
Lagrange’s theorem states that ....
A group is simple if ....

2. Suppose \( H \) and \( K \) are subgroups of a finite group \( G \) and \( \gcd(|H|, |K|) = 1 \). Show that \( H \cap K = \{e\} \).

3. For each element of \( \mathbb{Z}/12\mathbb{Z} \), determine its order.

4. Suppose \( \psi : G \to J \) is a group homomorphism and \( Q = \text{im}(\psi) \subseteq J \) is the image of \( \psi \). Let \( H = \ker(\psi) \).
   (a) Prove that \( H \) is a normal subgroup of \( G \).
   (b) Prove that the map \( \Psi : G/H \to Q \) defined by \( \Psi(gH) = \psi(g) \) is a well-defined map, i.e. if \( aH = bH \) for \( a, b \in H \), then \( \psi(a) = \psi(b) \) (giving \( \Psi(aH) = \Psi(bH) \)).
   (c) Prove that the map \( \Psi \) from (b) is a homomorphism.
   (d) Prove that \( \Psi : G/H \to Q \) is an isomorphism.

5. Suppose \( m \geq 1 \) is a positive integer. Prove that the subgroup \( H = m\mathbb{Z} \) of the group \( \mathbb{Z} \) (under addition) has index \( m \).

6. Suppose \( H \) and \( K \) are subgroups of a group \( G \). Recall that \( HK = \{hk|h \in H, k \in K\} \) and \( KH = \{kh|k \in K \text{ and } h \in H\} \). Show that \( HK \) is a subgroup of \( G \) if and only if \( HK = KH \).

7. In the group \( S_4 \), let \( V = \{(1), (12)(34), (13)(24), (14)(23)\} \).
   (a) Show that \( V \) is a subgroup of \( S_4 \).
   (b) What is \([S_4 : V]\)? List the cosets of \( V \) explicitly.
   (c) Show that \( V \) is a normal subgroup of \( S_4 \).
   (d) Since \( V \) is a normal subgroup of \( S_4 \), \( S_4/V \) has a natural group structure; calculate the coset \( (12)V \ast (123)V \).
(e) Show that \( W = \{ (1), (12)(34) \} \) is a normal subgroup of \( V \).
(f) Show that \( W \) is not a normal subgroup of \( S_4 \).

Remark. Note that \( W \) is normal in \( V \) and \( V \) is normal in \( S_4 \), but \( W \) is not normal in \( S_4 \). Thus, normality is not transitive.

8. If \( G \) is a group and \( H \) is a subgroup of \( G \) of index 2, then \( H \) is a normal subgroup of \( G \).

9. Prove that if \( G \) is a finite group, and \( Q \) is a homomorphic image of \( G \), then \(|Q| \) divides \(|G|\).

10. Prove that if \( G \) is a group of prime order \( p \), then \( G \) is cyclic.

11. Suppose \( G \) is a group of order 36 and \( K \) is a group of order 48, and that \( f : G \to K \) is a homomorphism. Let \( H = \ker(f) \). We obviously have \( 1 \leq |H| \leq 48 \). Which of these numbers cannot occur as \(|H|\)?

12. Suppose \( G \) is a group, \( x, y \in G \) are conjugate elements in \( G \), i.e. there exists \( g \in G \) such that \( y = gxg^{-1} \). Prove that \( x, y \) have the same order in \( G \).