Complex Numbers

1. (There are no zero-divisors in $\mathbb{C}$). Show that if $z, w \in \mathbb{C}$, and $zw = 0$ then either $z = 0$ or $w = 0$. (you may use the fact that this is true for $z, w \in \mathbb{R}$).

2. (a) (Every non-zero complex number is invertible). Show that for each $z \in \mathbb{C}$ such that $z \neq 0$, there exists a unique $w \in \mathbb{C}$ such that $wz = 1$, so it’s okay to write $w = z^{-1} = 1/z$.
   (b) Use (a) to give another proof of the statement in Problem 9.
   (c) For $z = 3 + 4i$, determine $1/z$ and write it in the form $a + bi$ with real numbers $a, b$.

3. (a) Show that for $z \in \mathbb{C}$, $z = 0$ if and only if $|z| = 0$.
   (b) Prove that $|zw| = |z||w|$.
   (c) Prove using induction that for all $n \in \mathbb{Z}$, $|z^n| = |z|^n$.

4. (a) Show that for $z, w \in \mathbb{C}$, $|z - w|$ is the usual distance from $z$ to $w$.
   (b) (Triangle Inequality) Give an algebraic proof of the fact that for $z, w \in \mathbb{C}$, $|z - w| \leq |z| + |w|$ and interpret this fact geometrically. Hint: First prove that if $u \in \mathbb{C}$, then $\Re(u) \leq |u|$. Next, argue that it suffices to show that $|z - w|^2 \leq (|z| + |w|)^2$. Now justify each step in the following:

$$|z - w|^2 = (z - w)(\overline{z} - \overline{w}) = |z|^2 + |w|^2 + 2\Re(-zw) \leq |z|^2 + |w|^2 + 2|z\overline{w}| = (|z| + |w|)^2.$$  

(c) Shade in the region $\{z \in \mathbb{C} \mid 1 \leq |z - i| \leq 2\}$. It is called an “annulus.” Hint: $|z - i|$ is the distance from $z$ to $i$.

5. (a) Find four solutions in $\mathbb{C}$ of the equation $z^4 = 1$.
   (b) Using your vast knowledge of trigonometry, evaluate $\zeta = \cos(\theta) + i\sin(\theta)$ where $\theta = 2\pi/6$.
   (c) Verify that $1, \zeta, \zeta^2, \zeta^3, \zeta^4, \zeta^5$ are six distinct solutions of $z^6 = 1$. They are called the sixth roots of unity in $\mathbb{C}$.
   (d) Draw a fairly accurate picture of the unit circle showing that the roots of $z^4 = 1$ and $z^6 = 1$ all lie on it. (Label the solutions). Use red for the 4 solutions of one equation and Blue for the six solutions of the other.

6. (Autour le théorème de De Moivre) For $z = r(\cos(\theta) + i\sin(\theta)) \in \mathbb{C}$, prove using induction on $n$ that for all $n \in \mathbb{Z}$, $z^n = r^n(\cos(n\theta) + i\sin(n\theta))$.

Extra Credit Problems.
1. Prove that the points \( z_1, z_2, z_3 \) in the complex plane are vertices of an equilateral triangle if and only if
\[
z_1^2 + z_2^2 + z_3^2 = z_1z_2 + z_1z_3 + z_2z_3.
\]

2. Let \( \zeta = e^{2\pi i/5} \) so that \( 1, \zeta, \zeta^2, \zeta^3, \zeta^4 \) are the vertices of a regular pentagon. The diagonals of this pentagon meet at the vertices of a smaller regular pentagon. Determine them.

3. (a) Show that for \( A \neq 0 \), the set of all points \( (x, y) \) in \( \mathbb{R}^2 \) satisfying \( Ax^2 + Ay^2 + Bx + Cy + D = 0 \) is either empty or a circle. Determine the center and the radius. What happens when \( A = 0 \)?

(b) Suppose \( z_1, z_2 \in \mathbb{C} \) are distinct fixed points in \( \mathbb{C} \) and \( K \) is a fixed positive real number, \( K \neq 1 \). Show that the set of all \( z \in \mathbb{C} \) satisfying
\[
\frac{|z - z_1|}{|z - z_2|} = K
\]
is a circle. Where is its center? What is its radius? How are \( z_1, z_2 \) positioned vis à vis this circle? If we keep \( K \) fixed and move \( z_1 \) along a straight line toward \( z_2 \), what happens to the center and radius of the circle? What happens when we move \( z_1 \) along the same straight line away from \( z_2 \)? If we keep \( z_1, z_2 \) fixed and move \( K \) toward 0 or toward \( \infty \), what happens to the circle? What happens when \( K = 1 \)?

4. (a) Let \( S \) be a set of size \( n \geq 1 \) and suppose \( r \) is an integer in the range \( 0 \leq r \leq n \). Let
\[
P_r(S) = \{ T \subseteq S \mid |T| = r \}
\]
be the set of all subsets of \( S \) of cardinality \( r \). Use the multiplication counting principle to deduce that
\[
|P_r(S)| = \frac{n!}{r!(n - r)!}.
\]
This number is often denoted by \( \binom{n}{r} \).

(b) With the above notations for \( n \) and \( r \) and for variables \( x \) and \( y \), derive the binomial formula
\[
(x + y)^n = \sum_{r=0}^{n} \binom{n}{r} x^r y^{n-r}.
\]

5. (a) Use the well-ordering principle to prove the Principle of Double Induction: Suppose for each pair \( (a, b) \in \mathbb{N} \times \mathbb{N} \), we have a statement \( P(a, b) \). Suppose i) \( P(1, 1) \) is true, and ii) Whenever \( P(k, l) \) true for some \( (k, l) \in \mathbb{N} \times \mathbb{N} \), then \( P(k + 1, l) \) and \( P(k, l + 1) \) are also true. Then \( P(a, b) \) is true for all \( (a, b) \in \mathbb{N} \).

(b) Now prove a slight modification: Suppose for all integers \( n, r \geq 1 \) with \( r \leq n \), we have a statement \( P(n, r) \). Suppose i) \( P(1, 1) \) is true and ii) Whenever \( P(k, l) \) is true for some \( (k, l) \in \mathbb{N} \times \mathbb{N} \) with \( l \leq k \), then \( P(k + 1, l) \) and \( P(k + 1, l + 1) \) are true. Then \( P(a, b) \) is true for all \( (n, r) \in \mathbb{N} \) with \( r \leq n \).

6. For a positive integer \( n \), we let \( I_n = \{ k \in \mathbb{Z} \mid 1 \leq k \leq n \} \) be the set of integers from 1 to \( n \). If \( T \) is a subset of \( I_n \), let \( m_T \) be the least element of \( T \). For \( 1 \leq r \leq n \), let \( f(n, r) \) be
the average, over all subsets $T$ of $I_n$ of cardinality $r$, of $m_T$. Recalling from problem 4 above that there are $\binom{n}{r}$ subsets of cardinality $r$ in $I_n$, we have, therefore,

$$f(n, r) := \frac{1}{\binom{n}{r}} \sum_{T \subseteq I_n, |T| = r} m_T.$$ 

Prove that

$$f(n, r) = \frac{n + 1}{r + 1}.$$