You should read Part 5 of Farshid’s notes.

1. Problems from Farshid’s Brain

1. Consider the following relation on \( \mathbb{Z} \): if \( a, b \in \mathbb{Z} \), then \( a \sim b \) if and only if \( a \cdot b \) is even. Prove or Disprove: \( \sim \) defines an equivalence relation on \( \mathbb{Z} \).

2. Suppose \( X \) is a set and \( \sim \) is an equivalence relation on \( X \). Suppose \( x, z \in X \). Prove that either \( \text{Eq}(x) = \text{Eq}(z) \) or else \( \text{Eq}(x) \cap \text{Eq}(z) = \emptyset \). [Hint: In other words, show that \( \text{cl}(x) \neq \text{cl}(z) \Rightarrow \text{cl}(x) \cap \text{cl}(z) = \emptyset \).]

3. Suppose \( \sim \) is an equivalence relation on a set \( X \) with graph \( R \). For \( x \in X \), show that \( R_x \cdot = R_{\cdot x} \).

4. If \( X \) is a set and \( \sim \) is an equivalence relation on it, then we have a map \( X \to X/\sim \) defined by \( x \mapsto \text{cl}(x) \). Show that this map is surjective. (Hint: this is a very easy problem; it requires only a careful examination of the definitions involved).

5. Suppose \( \Delta \subseteq \mathcal{P}(X) \setminus \emptyset \) is a collection of non-empty subsets of \( X \). Show that \( \Delta \) is a partition of \( X \) if and only if for every \( x \in X \) there exists a unique \( S \in \Delta \) such that \( x \in S \).

6. Consider the map \( f : \mathbb{Z} \to \mathbb{Z}_{12} \) where \( \mathbb{Z}_{12} = \{0, 1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11\} \) given by “reducing modulo 12” i.e. \( f(n) \) is the remainder of \( n \div 12 \).
   (a) Is this a surjective map? Explain.
   (b) Describe how this map induces a partition of \( \mathbb{Z} \).
   (c) Describe how this map induces an equivalent relation on \( \mathbb{Z} \) and give the defining rule for this relation, namely if \( x, y \in \mathbb{Z} \), then \( x \sim y \) if and only if ..........?
   (d) How many equivalence classes does this equivalence relation have?