1. Introductory Notes

1.1. Propositions. Much of mathematics is about solving problems. The process of doing mathematics is, however, far more complex than “simply” solving problems. For one thing, you have to create the problems in the first place! One of the things I would like to show you in this course is how mathematicians create mathematical objects and tools in order to solve specific problems. In the process of doing so, the objects and tools created bring forth new questions and new problems, from which new objects and tools blossom.

But, for now, to whet your appetite in this first week of class, let us start with the more familiar territory of problems: Do keep in mind that there will be plenty of “theory-building” coming later. Think of the problems below as puzzles, for that is what they are. Have fun with them. Examine their different components and how they fit together, make observations, jot down notes, locate and investigate patterns, try easier versions of the problem, relate the given problem to others you may have worked on, act the problem out, “see” the problem as a movie in your head, draw, doodle, sketch pictures, make graphs and tables and charts.

You will probably find it helpful to keep in mind the 4 steps George Polya advocates in approaching a problem:

i) Read and Understand the Problem
ii) Dream up a plan to solve the problem
iii) Carry out the plan
iv) Look back, returning to i) if necessary.

In some of the problems below (and throughout the semester, heck, throughout your life), you will be asked to provide a “proof” of some “Proposition.” Let us define how we will use these terms.

A Proposition is a common name for a mathematical statement in which all concepts that appear have been given a valid definition. A Proposition may be true or false. A Proof of a Proposition is an argument which establishes the truth of the Proposition. Here is an alternative description: a proof of $P$ is a convincing argument which removes any doubt concerning the truth of $P$. If a Proposition is true, we say that it holds. If a Proposition is false, we say that it does not hold.

Here are some examples of Propositions.

$P_1$: The sum of two even integers is even.

[What is meant here is shorthand for the following more precise version: Let us define our terms first: the integers are the elements of the familiar set

$$Z = \{\ldots, -3, -2, -1, 0, 1, 2, 3, \ldots\}$$]
on which we are given the usual operations of addition and multiplication. An integer \( a \) is even if there exists an integer \( m \) such that \( a = 2m \). Now here, finally, is the meaning of our statement: If \( a \) is an arbitrary even integer, and \( b \) is an arbitrary even integer, then \( a + b \) is an even integer. Is \( P1 \) true? Can you construct a proof of \( P1 \) for yourself?]

\[ P2: \) Every triangle is isosceles. [Although this is a very good example of a \textit{patently false} proposition, assuming the notion of distance between two points is the usual one, there is a very important class of distance-measurements (called \( p \)-adic distance, where \( p \) is a prime) for which the statement is true! If you campaign long and hard, say one second, then I will be glad to tell you more about \( p \)-adic metrics.]

\[ P3: \) If \( a \) and \( b \) are integers and \( b \neq 0 \), then there exists exactly one ordered pair of integers \((q, r)\) such that \( a = bq + r \) and \( 0 \leq r \leq |b| - 1 \). [See if you can figure out what commonly grasped mathematical process can convince you of the truth of this abstract-sounding statement!]

\[ P4: \) Every dog will have his day.

The status of the statement \( P4 \) as a Proposition is dubious, unless we give precise definitions to all the terms involved. In any case, the establishment of such a Proposition is outside the scope of mathematics.

1.2. Deductive Reasoning. Suppose you somehow establish the following facts.

- A. Every cat named Tom is grey.
- B. Garret has a cat named Tom.

I am reasonably certain you have already established in your mind a third fact that follows from these two, namely that Garret has a grey cat. We say that the general statement \( A \) and the particular statement \( B \) together imply
- C. Garret has a grey cat.

This an example of \textit{Deductive Reasoning}. That is to say, if you believe Statement A \textbf{and if} you believe Statement B, then you must necessarily believe Statement C. Note that whether or not you actually believe Statement A or Statement B is irrelevant to this discussion. The key idea being illustrated here is the validity of “implication arrow” \( A + B \Rightarrow C \), not the validity of either A or B or C on its own. What we are establishing is that you cannot logically believe statements A and B to be true without also believing C to be true. The notation \( P \Rightarrow Q \) is read “\( P \) implies \( Q \)” or “\( Q \) follows (logically) from \( P \),” or “If \( P \), then \( Q \)” (meaning “If \( P \) holds, then \( Q \) holds.”) or “\( Q \) is an implication of \( P \).” Okay that’s enough.

In Mathematics, we begin with \textit{axioms}, statements which we accept as true. We also give precise definitions for a variety of mathematical objects which, through much experience, we have decided recur often and deserve a name of their own and worthy of study. Our job is then to seek out true statements which follow logically from (i.e. are implied by) the axioms and the definitions we have given. How to choose these axioms and definitions is dictated in a natural way by the historical evolution of mathematics itself. And whether a given true statement about these objects is “interesting” or “useful” is also historically judged. You can be sure that the objects, axioms, concepts and theorems you encounter as a student of mathematics have “paid their dues” through hundreds and thousands of years of selective pressure; if they are still around, it means they are interesting and important.

Let me repeat: Our jobs as mathematicians is to seek out interesting statements which follow logically from (i.e. are implied by) the axioms and the definitions we have given. The statements that we establish to be logical consequences of the axioms are usually called
“Propositions,” the more important ones get dubbed “Theorems.” When a Theorem has interesting consequences which follow without much difficulty from the theorem, we call the resulting propositions a “Corollary” of the theorem. When we need an intermediary or auxiliary result on the way toward proving a Proposition, we call that a “Lemma.” Certain Lemmas become so indispensable, they become the mathematical equivalent of a screwdriver (the tool, not the drink, where is your mind?). It’s great to prove a theorem that becomes famous and gets your name attached to it, but many mathematicians secretly pine for having a Lemma of theirs that becomes famous. You can decide for yourself: would you rather be known as the inventor of the screwdriver (the tool not the drink!) or the inventor of the HydroMagneticDestabilizingMultiChannelTransmogrifier? On an even more basic level, when we are plodding along trying to solve a problem having to do with one set of objects, over many years and much trial and error, a collection of properties and ideas may coalesce together and inspire us to define a new mathematical object. Sometimes these objects then are so fascinating on their own, that they become the thing many people want to study. How satisfying do you think that might be?

1.3. Inductive Reasoning. Before I say anything about Inductive Reasoning itself, let me say that later in the course we will discuss a very important Deductive Reasoning Tool known as “Mathematical Induction,” and that Mathematical Induction and Inductive Reasoning are NOT to be confused with each other. Inductive Reasoning produces a statement that may or may or not be true. Mathematical Induction is a useful method for proving certain kinds of statements.

The term inductive reasoning refers to generalization based on observed patterns. It represents an “educated guess.” It is used in all the sciences, as well as in everyday life.

Here are some examples:

1 = 1^2 is a perfect square; 1 + 3 = 4 = 2^2 is a square; 1 + 3 + 5 = 3^2 is a square, 1 + 3 + 5 + 7 = 4^2 is a square; 1 + 3 + 5 + 7 + 9 = 5^2 is a perfect square! Wow, (here comes the observation): when we take a small bunch of odd numbers in order (starting with 1) and add them up, we get a perfect square. Dude, maybe, (here comes the generalization) no matter how many odd numbers you take (starting with 1 and going in order) and add them up, you always get a perfect square! This is inductive reasoning. The next step in the inductive reasoning process is to test a few more cases to see if the pattern continues to hold: 1 + 3 + 5 + 7 + 9 + 11 is just (1 + 3 + 5 + 7 + 9) + 11, i.e. the previous number plus 11, i.e. 25 + 11 = 36. If we add the next odd prime we get 36 + 13 = 49 which is the next square! If we add 49 + 15 = 64 it’s the next square. Dude, this seems like no accident. There has got to be a “reason” behind this. If you feel that way too, you are thinking like a mathematician already. Later in the course, we will discuss how Mathematical Induction can be used to prove the following more precise version of our observation:

Proposition. For any integer \( n \geq 1 \), the sum of the first \( n \) odd numbers is \( n^2 \).

2. Problems

NOTE: The first thing you should note is that there are very few problems. Nonetheless, get started right away, because it will take you a long time to come up with the solutions and write them up carefully. One of the goals of this course is to help you express your arguments cogently, concisely, and correctly (“the three co’s”). This is much harder than it sounds. A certain number of points for the assignment go toward each of the

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HOMEWORK 1: PROBLEM SOLVING, INDUCTIVE VS. DEDUCTIVE REASONING, AN INTRODUCTION TO PROOFS

co’s. Expect to suck at the three co’s for a while, but if you keep working at it and study the style of the arguments you encounter throughout this course, you will improve steadily. By the way, as I am lecturing, feel free to rate my proofs on the three co’s. You might have already guessed that I suck at “concisely.” Let me be totally up-front about this: sometimes in class, I will present proofs that suck on purpose, even in correctness. Your job is to catch those instances. Why do I do this? Sometimes it helps to see someone else fall into a ditch in order to learn how to avoid one yourself.

1. Suppose you are given eight identical-looking and identical-feeling balls. You are also given a balance scale. (By placing balls on the two pans of the balance, you can establish whether one collection of balls weighs the same as another collection of balls or not.) You are given the information that of the 8 balls, 7 of them weigh exactly the same and one of them is slightly heavier than the others. You are allowed to use the balance twice, no more.

Proposition. There is a strategy guaranteed to reveal the identity of the heavy ball in just two weighings.

Determine the truth or falsity of this Proposition. If you believe it is true, describe a strategy and explain exactly why it is guaranteed to work. If you do not believe it to be true, then give a convincing argument for the lack of a strategy.

2. Anna, Laura, Garret, and Aaron have been hired as coaches for basketball, soccer, volleyball, and swimming. Aaron, whose sister was hired to coach basketball, has never heard of Mia Hamm. Laura doesn’t like sports that involve balls. Which sport was each person hired to coach? Be sure to give a detailed and careful explanation of the reasoning process by which you arrived at your conclusion.

Note. In case you are not sure how Mia Hamm fits into the picture: in this problem, you can assume that anyone who has never heard of Mia Hamm is not the soccer coach.

3. (a) Consider the pattern formed by the sums of the first $n$ integers.

\[
\begin{align*}
1 &= 1 \\
1 + 2 &= 3 \\
1 + 2 + 3 &= 6 \\
1 + 2 + 3 + 4 &= 10 \\
\end{align*}
\]

Try to explain with a picture why the numbers 1, 3, 6, 10, 15, ... are called triangular numbers. [My own pet name for them is “bowling” numbers – can you explain that too?]
(b) The first bowling number is 1. Can you predict the tenth bowling number? Make a table of the first twelve bowling numbers. Can you determine hundredth bowling number? [When Gauss was six, he did this in a few seconds, astounding (and perhaps annoying?) his teacher.] Can you do it without writing them all down up to the hundreth? By messing around with a table of some bowling numbers, see if you can come up with a formula for the $n$th bowling number. I’m not asking you to prove the formula is correct, but if you want to do that, give it a whirl!

(c) Twenty people come to a fancy dinner. Each guest shakes hands with every other guest exactly once. How many handshakes occur?

(d) Twenty points are marked on the perimeter of a circle. The line segment joining every pair of distinct marked points is drawn. How many line segments are drawn? (Try it as a thought experiment or as an actual activity).

(e) Now the number of people coming to the fancy dinner keeps fluctuating, so we just want to call the number of guests $x$ and wish to have a formula $H(x)$, where $H(x)$ is the number of handshakes among $x$ guests. What about where $H(x)$ is the number of line segments among $x$ marked points on the circle?

4. Aaron Wolbach has 44 pennies and ten pockets in his vest. He wants to put all his pennies in his pocket in such a way that he has a different number of pennies in each pocket. Can he do it? If yes, tell Aaron how, if not, explain to him (gently) why he cannot.

5. Consider a clock with line-thin hour-hand and minute-hand that move perfectly continuously. At noon and then again at midnight, the two hands line up perfectly.

(a) How many times between noon and midnight (counting midnight, but not noon) do the two hands line up again?

(b) Determine the exact times at which the two hands line up. Explain your reasoning very carefully.

3. Extra Credit Problems.

AN IMPORTANT NOTE ABOUT EXTRA CREDIT PROBLEMS. These problems are for your amusement and edification and to challenge or push you. Don’t look here for an easy way to ameliorate your grade: a better way of doing that would be to concentrate more on other aspects of the course. There is no expectation here about whether you “should” be able to solve even a small part of any of these problems. In some cases, the solutions here are difficult and beyond this course entirely. In some cases, I do not myself know the solution to the problems stated here, and in fact the solution may be as yet unknown to homo sapiens. And in other cases, the solution may come to you easily, in a flash. The point here is to present what I think are interesting problems and see how far you can run with them, for your own entertainment and growth. You will receive a certain number of “bonus points” depending on the difficulty of the problem and how far you travelled into the solution. At the end of the semester, the student with the largest number of bonus points will receive a fabulous (sur)prize. I will be reluctant to reveal too many solutions too soon, as I want people time to hammer away at their favorites for quite some time.
A. Consider a generalization of Problem 1. Namely you are given \( n \) balls, 1 of which weighs slightly more than all the others, while the other \( n - 1 \) are identical. Let \( s(n) \) be the smallest number \( k \geq 0 \) with the following property: if you are allowed to use the balance scale \( k \) times, then there is a strategy that will always allow you to identify the heavy ball.

For instance, in Problem 1, the Proposition states that \( s(8) \leq 2 \). Perhaps you will find it easy to convince yourself that \( s(8) > 1 \). Therefore, if the Proposition is true, then \( s(8) = 2 \).

Now, finally, here is the question: Can you find an explicit formula (or a procedure for determining) \( s(n) \) as a function of \( n \)? Short of that, can you prove any bounds on this function? For example, is it true that \( s(n) \leq n \)? How about \( s(n) \leq n/2 \)? How about \( s(n) \leq n/3 \)?

Here are some strategies that may be useful (or they might just distract you from the true path, depending on your style of thinking!): make a chart of \( n \) versus \( s(n) \) for small values of \( n \). If you are not sure what \( s(n) \) is exactly, at least record a guess or a bound. Turn the problem on its head: Say you allow yourself \( k \) uses of the balance and let \( N(k) \) be the largest number of balls you can start with and still have a strategy guaranteed to reveal the one heavy ball via \( k \) weighings. If you can determine \( N(k) \), you should be able to use that easily to determine \( s(n) \).

B. Now let us generalize Problem 1 even a bit more. Let \( 1 \leq h \leq n \) be two whole numbers. Suppose you are given \( n \) balls, and the balls are of two types: \( h \) of the balls are slightly heavier (but all identical to each other) and the remaining \( n - h \) are all identical to each other and slightly lighter than the other \( h \). Now let \( S(n, h) \) be the least number of uses of the balance scale that are needed in order to guarantee that you will be able to distinguish the heavy balls from the light. In particular, \( S(n, 1) \) is just the \( s(n) \) of Problem A.

The ultimate question is the same as before: Determine \( S(n, h) \) exactly if you can, but short of that, provide some bounds for it.

Start by determining \( S(n, 2) \) or bounds on it.

Have fun! Experiment, act it out, make charts, wild guesses, test your guesses, ask roommates for their crazy ideas..... Can you see a relationship between \( S(n, h) \) and \( S(n, n - h) \)? Is this useful at all? (Maybe it is and maybe it isn’t).