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On The Fontaine-Mazur Conjecture

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I. The Fontaine-Mazur Conjecture

$K$ a number field, signature $(r_1, r_2)$
$G_K = \text{Gal}(\bar{K}/K)$

An irreducible $p$-adic Galois representation
$\rho : G_K \to \text{GL}_n(\mathbb{Q}_p)$ is called geometric if

- $\rho$ is unramified outside a finite set of places of $K$
- $\rho$ is potentially semistable at all places

Remarks 1. More precisely, the second condition requires that the restriction of $\rho$ to each decomposition group $D_v$ become semistable (in the sense of Fontaine) after a suitable finite base change.

2. By a Theorem of Grothendieck, at a place $v$ of residue characteristic $\ell \neq p$, every representation is potentially semistable (after base change, its image is quasi-unipotent).
We say \( \rho \) comes from \textit{algebraic geometry} if it is a Tate twist of the action of \( G_K \) on a subquotient of the étale cohomology of some (smooth, projective) variety over \( K \).

\textbf{CONJECTURE (FM, '95)} \( \rho \) is geometric \( \iff \) it comes from algebraic geometry.

The “\( \iff \)” implication has a much longer history and was established by Tsuji '99. The other direction is a vast “Reciprocity Law,” e.g. FM implies that elliptic curves over \( \mathbb{Q} \) are modular: FM \( \Rightarrow \) STWWTDCB.
II. Tamely Ramified Fontaine-Mazur

Let $S$ be a finite set of places of $K$. If the places in $S$ all have residue characteristic $\neq p$ we say that “$S$ is away from $p$”

- $K_S = \text{the maximal } p\text{-extension (inside } \bar{K} \text{) of } K \text{ unramified outside } S$
- $G_S = \text{Gal}(K_S/K)$, a finitely generated pro-$p$ group

**Tame-Wild Dichotomy**

- **Wild Case** If $S$ contains all the $K$-primes dividing $p$, then $G_S$ is infinite: e.g. $G_S$ is equipped with (at least $r_2 + 1$) surjections to $\mathbb{Z}_p$, and these go a long way toward illuminating the structure of $G_S$ (e.g. it has finite cohomological dimension, ...)

- **Tame Case** On the other hand, if $S$ is away from $p$, then sometimes $G_S$ is finite, and sometimes it is infinite (as first shown by Golod and Shafarevich in 1964).
Notation. If $G$ is a pro-$p$ group, $d(G) = \dim_{\mathbb{F}_p} H^1(G, \mathbb{F}_p)$, $r(G) = \dim_{\mathbb{F}_p} H^2(G, \mathbb{F}_p)$. These are the minimal number of generators and relations, respectively, in a presentation of $G$ as a pro-$p$ group. Note that $d(G) = d(G^{\text{ab}})$ (Burnside Basis Thm); in particular, we can calculate $d(G_S)$ as the rank of an appropriate class group by class field theory.

Golod-Shafarevich Strategy ($S$ away from $p$):

A. $r(G_S) - d(G_S) \leq r_1 + r_2 \leq [K : \mathbb{Q}]$

B. If $G$ is a finite $p$-group, then $r(G) > d(G)^2/4$.

$A + B = C$. If $d(G_S) \geq 2 + 2\sqrt{r_1 + r_2 + 1}$, then $G_S$ is infinite.

Other than the Golod-Shafarevich Criterion above, not much is known about the “tame fundamental groups” $G_S$. 
The Fontaine-Mazur conjecture has the following consequence for the structure of $G_S$ in the tame case, assuming standard alg. geom. conjectures (Tate, Hodge):

**CONJECTURE (TAME FM).** If $S$ is away from $p$, then any representation $G_S \to \text{GL}_n(\mathbb{Z}_p)$ has finite image, i.e. $G_S$ has no infinite $p$-adic analytic quotient.

More generally,

**CONJECTURE (Boston).** If $S$ is away from $p$, and $\Lambda$ is a complete local Noetherian ring with residue field of characteristic $p$, then any representation $G_S \to \text{GL}_n(\Lambda)$ has finite image.

Some cases of this have been verified by Boston.
III. Shallow Ramification: A Variant on Tame

- $S$ a finite set of places of $K$

- $\nu : S \rightarrow \mathbb{R}$ an arbitrary indexing map

Let $K_{S,\nu}$ be the compositum of all finite $p$-extensions $L/K$ such that every higher ramification group $D^{\nu p}(L/K, p)$ is trivial (for all $p \in S$). (Ramification groups in the upper numbering behave well under quotients). The ramification in $K_{S,\nu}/K$ is “shallow,” because the “depth” of ramification (terminology due to Coates-Greenberg) is bounded by the indexing function $\nu$.

Let $G_{S,\nu} = \text{Gal}(K_{S,\nu}/K)$. Thus, $G_{S,\nu}$ = the quotient of $G_S$ by the closed normal subgroup generated by all higher ramification groups $D^{\nu p}(K_S/K, p)$ as $p$ runs over $S$. 
Since \( p \) is tamely ramified in \( L \) if and only if \( D^1(L/K, p) \) vanishes, \( G_S = G_{S,\nu} \) in the case where \( S \) contains no primes of residue characteristic \( p \).

We know very little about \( G_{S,\nu} \). For instance,

**Problem.** Give an upper bound for the partial Euler characteristic \( r(G_{S,\nu}) - d(G_{S,\nu}) \).

Our general philosophy is that **shallow ramification is just like tame ramification**. In particular, we expect

**CONJECTURE (SHALLOW FM).** If \( \nu \) is finite, then every representation \( \rho : G_{S,\nu} \to \text{GL}_n(\mathbb{Z}_p) \) has finite image.

In fact,

**TAME FM \Rightarrow SHALLOW FM**
by the following generalization of Grothendieck’s theorem:

**THEOREM.** If $\rho : G_K \to \text{GL}_n(\mathbb{Z}_p)$ has shallow ramification, i.e. factors through some $G_{S,\nu}$, then $\rho$ is everywhere potentially semistable.

Idea of proof: By a theorem of Sen, in a $p$-adic analytic totally ramified extension of $p$-adic fields, the filtration by higher ramification groups is intertwined with the $p$-adic Lie filtration. Therefore, if the depth of ramification is bounded, the inertia group is finite. Applying this to each prime of residue characteristic $p$ in $S$, we find that $\rho$ is potentially tamely ramified, then appeal to Grothendieck.

**IV. Root Discriminants**

We can characterize shallow vs. deeply ramified extensions in terms of root discriminants as follows.
Root Discriminant $rd_K = |d_K|^{1/n}$ where

$$n = [K : \mathbb{Q}], \quad d_K = \text{discriminant of } K.$$ 

If $L/K$ is an infinite algebraic extension, we say $L/K$ is **asymptotically bad** if there is a sequence of number fields $K \subset K_1 \subset K_2 \subset \cdots \subset L$ with $rd_{K_i} \to \infty$. Otherwise, we say $L/K$ is **asymptotically good**. The traditional example of an asymptotically good tower is an unramified one, where the root discriminant is constant.

**PROPOSITION.** An extension (unramified outside a finite set) with shallow ramification is asymptotically good. Indeed, if $m = [K : \mathbb{Q}]$, and $K \subseteq L \subseteq K_{S,\nu}$, then

$$rd_L \leq rd_K \prod_{p \in S} N_{K/\mathbb{Q}}(p)^{(\nu_p+1)/m}.$$ 

If $L/K$ is deeply ramified, then it is asymptotically bad.
The extension of Grothendieck’s theorem from tame to shallow ramification has the following consequence.

**THEOREM.** Conjecture TAME FM is equivalent to the following statement:

If $L/K$ has infinite $p$-adic analytic Galois group, then $L/K$ is asymptotically bad.

**Proof:** Assume TAME FM. If $L/K$ is ramified at infinitely many primes, then $L/K$ is automatically asymptotically bad. If $L/K$ is ramified at finitely many primes, then by TAME FM, it must be deeply ramified hence bad. On the other hand, supposing every infinite $p$-adic analytic extension of $K$ is bad, and knowing that an infinite tame extension unramified outside a finite set is asymptotically good, we would conclude that $K$ admits no infinite analytic tame extension unramified outside a finite set of primes.