The homology of tropical varieties

Paul Hacking

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1 Introduction

Given a closed subvariety X of an algebraic torus T, the associated tropical variety is a polyhedral fan in the space of 1-parameter subgroups of the torus which describes the behaviour of the subvariety at infinity. We show that the link of the origin has only top rational homology if a genericity condition is satisfied. Our result is obtained using work of Tevelev [T] and Deligne's theory of mixed Hodge structures [D].

Here is a sketch of the proof. We use the tropical variety of X to construct a smooth compactification $X \subset \overline{X}$ with simple normal crossing boundary B. We relate the link L of the tropical variety to the *dual complex* K of B, that is, the simplicial complex with vertices corresponding to the irreducible components B_i of B and simplices of dimension j corresponding to $(j+1)$-fold intersections of the B_i. Following [D] we identify the homology groups of K with graded pieces of the weight filtration of the cohomology of X. Since X is an affine variety, it has the homotopy type of a CW complex of real dimension equal to the complex dimension of X. From this we deduce that K and L have only top homology.

The link of the tropical variety of $X \subset T$ was previously shown to have only top homology in the following cases: the intersection of the Grassmannian $G(3,6)$ with the big torus T in its Plücker embedding [SS], the complement of an arrangement of hyperplanes [AK], and the space of matrices of rank ≤ 2 in $T = (\mathbb{C}^{\times})^{m \times n}$ [MY]. We discuss these and other examples from our viewpoint in Sec. 4.

It has been conjectured that the link of the tropical variety of an *arbitrary* subvariety of a torus is homotopy equivalent to a bouquet of spheres (so, in particular, has only top homology). I expect that this is false in general, but I do not know a counterexample. See also Rem. 2.11.

We note that D. Speyer has used similar techniques to study the topology of the tropicalisation of a curve defined over the field $\mathbb{C}((t))$ of formal power series, see [S, Sec. 10].

2 Statement of Theorem

We work throughout over $k = \mathbb{C}$. Let $X \subset T$ be a closed subvariety of an algebraic torus $T \simeq (\mathbb{C}^\times)^r$. Let $K = \bigcup_{n \geq 1} \mathbb{C}((t^{1/n}))$ be the field of Puiseux series (the algebraic closure of the field $\mathbb{C}((t))$ of Laurent series) and $\mathrm{ord}\colon K^\times \to \mathbb{Q}$ the valuation of $K/\mathbb{C}$ such that $\mathrm{ord}(t) = 1$.

Let $M = \mathrm{Hom}(T, \mathbb{C}^\times) \simeq \mathbb{Z}^r$ be the group of characters of T and $N = M^*$. We have a natural map

$$\mathrm{val}\colon T(K) \to N_{\mathbb{Q}}$$

given by

$$T(K) \ni P \mapsto (\,\chi^m \mapsto \mathrm{ord}(\chi^m(P))\,).$$

In coordinates

$$(K^\times)^r \ni (a_1, \ldots, a_r) \mapsto (\mathrm{ord}(a_1), \ldots, \mathrm{ord}(a_r)) \in \mathbb{Q}^r$$

Definition 2.1. [EKL, 1.2.1] The *tropical variety* $\mathcal{A}$ of X is the closure of $\mathrm{val}(X(K))$ in $N_{\mathbb{R}} \simeq \mathbb{R}^r$.

Theorem 2.2. *[EKL, 2.2.5] $\mathcal{A}$ is the support of a rational polyhedral fan in $N_{\mathbb{R}}$ of pure dimension $\dim X$.*

Let Σ be a rational polyhedral fan in $N_{\mathbb{R}}$. Let $T \subset Y$ be the associated torus embedding. Let $\overline{X} = \overline{X}(\Sigma)$ be the closure of X in Y.

Theorem 2.3. *[T, 2.3] $\overline{X}$ is compact iff the support $|\Sigma|$ of Σ contains $\mathcal{A}$.*

From now on we always assume that $\overline{X}$ is compact.

Theorem 2.4. *[ST, 3.9][T2] The intersection $\overline{X} \cap O$ is non-empty and has pure dimension equal to the expected dimension for every torus orbit $O \subset Y$ iff $|\Sigma| = \mathcal{A}$.*

Proof. Suppose $|\Sigma| = \mathcal{A}$. We first show that $\overline{X} \cap O$ is nonempty for every orbit $O \subset Y$. Let $\Sigma' \to \Sigma$ be a strictly simplical refinement of Σ and $f\colon Y' \to Y$ the corresponding toric resolution of Y. Let $\overline{X}'$ be the closure of X in Y'. Let $O \subset Y$ be an orbit, and $O' \subset Y'$ an orbit such that $f(O') \subseteq O$. Then $\overline{X}' \cap O' \neq \emptyset$ by [T, 2.2], and $f(\overline{X}' \cap O') \subseteq \overline{X} \cap O$, so $\overline{X} \cap O \neq \emptyset$ as required.

We next show that $\overline{X} \cap O$ has pure dimension equal to the expected dimension for every orbit $O \subset Y$. Let $O \subset Y$ be an orbit of codimension c. Let Z be an irreducible component of the intersection $\overline{X} \cap O$ with its reduced induced structure. Let W be the closure of O in Y and $\overline{Z}$ the closure of Z in W. Then, since $\overline{Z}$ is compact, the fan of the toric variety W contains the tropical variety of $Z \subset O$ by Thm. 2.3. We deduce that $\dim Z \leq \dim X - c$ by Thm. 2.2. On the other hand, since toric varieties are Cohen-Macaulay, the orbit $O \subset Y$ is cut out set-theoretically by a regular sequence of length c at each point of O. It follows that $\dim Z \geq \dim X - c$, so $\dim Z = \dim X - c$ as required.

The converse follows from [ST, 3.9]. □

Here is the main result of this paper.

Theorem 2.5. *Suppose that $|\Sigma| = \mathcal{A}$ and the following condition is satisfied:*

(∗) *For each torus orbit $O \subset Y$, $\overline{X} \cap O$ is smooth and is connected if it has positive dimension.*

Then the link L of $0 \in \mathcal{A}$ has only top reduced rational homology, i.e., $\tilde{H}_i(L, \mathbb{Q}) = 0$ for $i < \dim L = \dim X - 1$.

Example 2.6. Let $\overline{Y}$ be a projective toric variety. Let $\overline{X} \subset \overline{Y}$ be a complete intersection. That is, $\overline{X} = H_1 \cap \cdots \cap H_c$ where H_i is an ample divisor on $\overline{Y}$. Assume that H_i is a general element of a basepoint free linear system for each i. Let $Y \subset \overline{Y}$ be the open toric subvariety consisting of orbits meeting $\overline{X}$ and Σ the fan of Y. Then $|\Sigma| = \mathcal{A}$ by Thm. 2.4 and $\overline{X} \subset Y$ satisifes the condition (∗) by Bertini's theorem [H, III.7.9, III.10.9].

If $\overline{\Sigma}$ is the (complete) fan of $\overline{Y}$, the fan Σ is the union of the cones of $\overline{\Sigma}$ of codimension $\geq c$. So it is clear in this example that the link L of $0 \in \mathcal{A}$ has only top reduced homology. Indeed, let $r = \dim Y$. Then the link K of $0 \in \overline{\Sigma}$ is a polyhedral subdivision of the $(r-1)$-sphere, and L is the $(r-c-1)$-skeleton of K, hence $\tilde{H}_i(L, \mathbb{Z}) = \tilde{H}_i(S^{r-1}, \mathbb{Z}) = 0$ for $i < r - c - 1$.

A useful reformulation of condition (∗) is given by the following lemma.

Lemma 2.7. *Assume that $|\Sigma| = \mathcal{A}$. Then the following conditions are equivalent.*

(1) $\overline{X} \cap O$ is smooth for each orbit $O \subset Y$.

(2) The multiplication map $m \colon T \times \overline{X} \to Y$ is smooth.

Proof. The fibre of the multiplication map over a point $y \in O \subset Y$ is isomorphic to $(\overline{X} \cap O) \times S$, where $S \subset T$ is the stabiliser of y. Now m is smooth iff it is flat and each fibre is smooth. The map m is surjective and has equidimensional fibres by Thm. 2.4. Finally, if W is integral, Z is normal, and $f\colon W \to Z$ is dominant and has reduced fibres, then f is flat iff it has equidimensional fibres by [EGA4, 14.4.4, 15.2.3]. This gives the equivalence. □

Definition 2.8. [T, 1.1,1.3] We say $\overline{X} \subset Y$ is *tropical* if $m\colon T \times \overline{X} \to Y$ is flat and surjective. (Then in particular $\overline{X} \cap O$ is non-empty and has the expected dimension for each orbit $O \subset Y$, so $|\Sigma| = \mathcal{A}$ by Thm. 2.4.) We say $X \subset T$ is *schön* if m is smooth for some (equivalently, any [T, 1.4]) tropical compactification $\overline{X} \subset Y$.

Example 2.9. Here we give some examples of schön subvarieties of tori. (For more examples see Sec. 4.)

(1) Let $\overline{Y}$ be a projective toric variety and $\overline{X} \subset \overline{Y}$ a general complete intersection as in Ex. 2.6. Let $T \subset \overline{Y}$ be the big torus and $X = \overline{X} \cap T$. Then $\overline{X} \cap O$ is either empty or smooth of the expected dimension for every orbit $O \subset \overline{Y}$ by Bertini's theorem. Hence $X \subset T$ is schön.

(2) Let $\overline{Y}$ be a projective toric variety and G a group acting transitively on $\overline{Y}$. Let $\overline{X} \subset \overline{Y}$ be a smooth subvariety. Then, for $g \in G$ general, $g\overline{X} \cap O$ is either empty or smooth of the expected dimension for every orbit $O \subset Y$ by [H, III.10.8]. Let $T \subset \overline{Y}$ be the big torus and $X' = g\overline{X} \cap T$. Then $X' \subset T$ is schön for $g \in G$ general.

Example 2.10. Here is a simple example $X \subset T$ which is not schön. Let $\overline{Y}$ be a projective toric variety and $\overline{X} \subset \overline{Y}$ a closed subvariety such that $\overline{X}$ meets the big torus $T \subset \overline{Y}$ and $\overline{X}$ is singular at a point which is contained in an orbit $O \subset \overline{Y}$ of codimension 1. Let $X = \overline{X} \cap T$. Then $X \subset T$ is not schön. Indeed, suppose that $m\colon T \times \overline{X}' \to Y'$ is smooth for some tropical compactification $\overline{X}' \subset Y'$. We may assume that the toric birational map $f\colon Y' \dashrightarrow \overline{Y}$ is a morphism by [T, 2.5]. Now $\overline{X} \cap O$ is singular by construction, and $f\colon Y' \to \overline{Y}$ is an isomorphism over O because $O \subset \overline{Y}$ has codimension 1, hence $\overline{X}' \cap f^{-1}O$ is also singular, a contradiction.

Remark 2.11. It has been suggested that the link L of the tropical variety of an *arbitrary* subvariety of a torus is homotopy equivalent to a bouquet of top dimensional spheres (so, in particular, has only top homology). I expect that this is false in general, but I do not know a counterexample.

However, there are many examples where the hypothesis $(*)$ of Thm. 2.5 is not satisfied but the conclusion is still valid. For example, let $\overline{X} \subset \overline{Y}$ be a complete intersection in a projective toric variety such that $\overline{X} \cap O$ has the expected dimension for each orbit $O \subset \overline{Y}$ and let $X = \overline{X} \cap T \subset T$ where $T \subset \overline{Y}$ is the big torus. Then $X \subset T$ is not schön in general but L is a bouquet of top-dimensional spheres, cf. Ex. 2.10, 2.6. See also Ex. 4.4 for another example.

Construction 2.12. [T, 1.7] We can always construct a tropical compactification $\overline{X} \subset Y$ as follows. Choose a projective toric compactification $\overline{Y}_0$ of T. Let $\overline{X}_0$ denote the closure of X in $\overline{Y}_0$. Assume for simplicity that

$$S = \{t \in T \mid t \cdot X = X\} \subset T$$

is trivial (otherwise, we can pass to the quotient $X/S \subset T/S$). Consider the embedding $T \hookrightarrow \text{Hilb}(\overline{Y}_0)$ given by $t \mapsto t^{-1}[\overline{X}_0]$. Let $\overline{Y}$ be the normalisation of the closure of T in $\text{Hilb}(\overline{Y}_0)$. (So $\overline{Y}$ is a projective toric compactification of T.) Let $\overline{X}$ be the closure of X in $\overline{Y}$, and $Y \subset \overline{Y}$ the open toric subvariety consisting of orbits meeting $\overline{X}$. Let $\mathcal{U} \subset \text{Hilb}(\overline{Y}_0) \times \overline{Y}_0$ denote the universal family over $\text{Hilb}(\overline{Y}_0)$ and $\mathcal{U}^0 = \mathcal{U} \cap (\text{Hilb}(\overline{Y}_0) \times T)$. One shows that there is an identification

$$\begin{array}{ccc} T \times X & \xrightarrow{\sim} & \mathcal{U}^0|_Y \\ & {\scriptstyle m}\searrow \quad \swarrow & \\ & Y & \end{array} \qquad (1)$$

given by $(t, x) \mapsto (tx, t)$ [T, p. 1093, Pf. of 1.7]. In particular, m is flat.

Remark 2.13. We note that, in the situation of 2.12, we can verify the condition $(*)$ using Gröbner basis techniques. Let $O \subset Y$ be an orbit. Let σ be the cone in the fan of Y corresponding to O, and $w \in N$ an integral point in the relative interior of σ. We regard w as a 1-parameter subgroup $\mathbb{C}^\times \to T$ of T. Then, by construction, the limit $\lim_{t \to 0} w(t)$ lies in the orbit O. Let $\overline{X}_0^w$ be the flat limit of the 1-parameter family $w(t)^{-1}\overline{X}_0$ as $t \to 0$. Then the fibres of $\mathcal{U} \to \text{Hilb}(\overline{Y}_0)$ over O are the translates of $\overline{X}_0^w$. Let $y \in O$ be a point and $S \subset T$ the stabiliser of y. The fibre of m over y is isomorphic to both $(\overline{X} \cap O) \times S$ and $\overline{X}_0^w \cap T$ (by the identification (1)). Hence $\overline{X} \cap O$ is smooth (resp. connected) iff $\overline{X}_0^w \cap T$ is so. Suppose now that $\overline{Y}_0 \simeq \mathbb{P}^N$, and let $I \subset k[X_0, \ldots, X_N]$ be the homogeneous ideal of $\overline{X}_0 \subset \mathbb{P}^N$. Then $\overline{X}_0^w$ is the zero locus of the initial ideal of I with respect to w.

3 The stratification of the boundary and the weight filtration

Let $\overline{X}$ be a smooth projective variety of dimension n, and $B \subset \overline{X}$ a simple normal crossing divisor. We define the *dual complex* of B to be the CW complex K defined as follows. Let $B_1, \ldots, B_m$ be the irreducible components of B and write $B_I = \bigcap_{i \in I} B_i$ for $I \subset [m]$. To each connected component Z of B_I we associate a simplex σ with vertices labelled by I. The facet of σ labelled by $I \setminus \{i\}$ is identified with the simplex corresponding to the connected component of $B_{I \setminus \{i\}}$ containing Z.

Theorem 3.1. *The reduced homology of K is identified with the top graded pieces of the weight filtration on the cohomology of the complement $X = \overline{X} \backslash B$. Precisely,*

$$\tilde{H}_i(K, \mathbb{C}) = \operatorname{Gr}^W_{2n} H^{2n-(i+1)}(X, \mathbb{C}).$$

Corollary 3.2. *If X is affine, then*

$$\tilde{H}_i(K, \mathbb{C}) = \begin{cases} \operatorname{Gr}^W_{2n} H^n(X, \mathbb{C}) & \text{if } i = n-1 \\ 0 & \text{otherwise} \end{cases}$$

Proof of Thm. 3.1. This is essentially contained in [D], see also [V, Sec. 8.4]. Define a filtration $\tilde{W}$ of the complex $\Omega^{\cdot}_{\overline{X}}(\log B)$ of differential forms on $\overline{X}$ with logarithmic poles along B by

$$\tilde{W}_l \Omega^k_{\overline{X}}(\log B) = \Omega^l_{\overline{X}}(\log B) \wedge \Omega^{k-l}_{\overline{X}}.$$

The filtration of $\Omega^{\cdot}_{\overline{X}}(\log B)$ yields a spectral sequence

$$E_1^{p,q} = \mathbb{H}^{p+q}(\overline{X}, \operatorname{Gr}^{\tilde{W}}_{-p} \Omega^{\cdot}_{\overline{X}}(\log B)) \implies \mathbb{H}^{p+q}(\Omega^{\cdot}_{\overline{X}}(\log B)) = H^{p+q}(X, \mathbb{C}).$$

which defines a filtration $\tilde{W}$ on $H^{\cdot}(X, \mathbb{C})$. The *weight filtration* W on $H^i(X, \mathbb{C})$ is by definition the shift $W = \tilde{W}[i]$, i.e., $W_j(H^i) = \tilde{W}_{j-i}(H^i)$. The spectral sequence degenerates at E_2 [D, 3.2.10], so

$$E_2^{p,q} = \operatorname{Gr}^{\tilde{W}}_{-p} H^{p+q}(X, \mathbb{C}).$$

The E_1 term may be computed as follows. Let $\tilde{B}^l$ denote the disjoint union of the l-fold intersections of the components of B, and j_l the map $\tilde{B}^l \to \overline{X}$. (By convention $\tilde{B}^0 = \overline{X}$.) The Poincaré residue map defines an isomorphism

$$\operatorname{Gr}^{\tilde{W}}_l \Omega^k_{\overline{X}}(\log B) \xrightarrow{\sim} j_{l*} \Omega^{k-l}_{\tilde{B}^l}, \tag{2}$$

see [V, Prop. 8.32]. This gives an identification

$$E_1^{p,q} = \mathbb{H}^{p+q}(\overline{X}, \operatorname{Gr}^{\tilde{W}}_{-p} \Omega^{\cdot}_{\overline{X}}(\log B)) = \mathbb{H}^{2p+q}(\tilde{B}^{(-p)}, \Omega^{\cdot}_{\tilde{B}^{(-p)}}) = H^{2p+q}(\tilde{B}^{(-p)}, \mathbb{C}).$$

The differential

$$d_1 \colon H^{2p+q}(\tilde{B}^{(-p)}) \to H^{2(p+1)+q}(\tilde{B}^{(-p-1)})$$

is identified (up to sign) with the Gysin map on components [V, Prop. 8.34]. Precisely, write $s = -p$. Then $d_1 \colon H^{q-2s}(\tilde{B}^{(s)}) \to H^{q-2(s-1)}(\tilde{B}^{(s-1)})$ is given by the maps

$$(-1)^{s+t} j_* \colon H^{q-2s}(B_I) \to H^{q-2(s-1)}(B_J),$$

where $I = \{i_1 < \cdots < i_s\}$, $J = I \backslash \{i_t\}$, j denotes the inclusion $B_I \subset B_J$, and j_* is the Gysin map. Equivalently, identify $H^{q-2s}(\tilde{B}^{(s)}) = H_{2n-q}(\tilde{B}^{(s)})$ by Poincaré duality. Then $d_1 \colon H_{2n-q}(\tilde{B}^{(s)}) \to H_{2n-q}(\tilde{B}^{(s-1)})$ is given by the maps

$$(-1)^{s+t} j_* \colon H_{2n-q}(\tilde{B}^{(s)}) \to H_{2n-q}(\tilde{B}^{(s-1)}).$$

So, the E_1 term of the spectral sequence is as follows.

$$\begin{array}{ccccccccc}
H_0(\tilde{B}^{(n)}) & \to & H_0(\tilde{B}^{(n-1)}) & \to & \cdots & \to & H_0(\tilde{B}^{(1)}) & \to & H_0(\tilde{B}^{(0)}) \\
 & & H_1(\tilde{B}^{(n-1)}) & \to & \cdots & \to & H_1(\tilde{B}^{(1)}) & \to & H_1(\tilde{B}^{(0)}) \\
 & & & & & & \vdots & & \vdots \\
 & & & & & & H_{2n-2}(\tilde{B}^{(1)}) & \to & H_{2n-2}(\tilde{B}^{(0)}) \\
 & & & & & & & & H_{2n-1}(\tilde{B}^{(0)}) \\
 & & & & & & & & H_{2n}(\tilde{B}^{(0)})
\end{array}$$

The top row ($q = 2n$) is the complex

$$\cdots \to H_0(\tilde{B}^{(s+1)}) \to H_0(\tilde{B}^{(s)}) \to H_0(\tilde{B}^{(s-1)}) \to \cdots,$$

which computes the reduced homology of the dual complex K of B. We deduce

$$\operatorname{Gr}^{\tilde{W}}_s H^{2n-s}(X, \mathbb{C}) = \tilde{H}_{s-1}(K, \mathbb{C}).$$

□

Proof of Corollary 3.2. If X is affine then X has the homotopy type of a CW complex of dimension n, so $H^k(X, \mathbb{C}) = 0$ for $k > n$. □

Proof of Thm. 2.5. By our assumption and Lem. 2.7 the multiplication map $m\colon T \times \overline{X} \to Y$ is smooth. Let $Y' \to Y$ be a toric resolution of Y given by a refinement Σ' of Σ. Then $m'\colon T \times \overline{X}' \to Y'$ is also smooth — it is the pullback of m [T, 2.5]. So $\overline{X}'$ is smooth with simple normal crossing boundary $B' = \overline{X}' \setminus X$ (because this is true for Y'). Hence the dual complex K of B' has only top reduced rational homology by Cor. 3.2.

It remains to relate K and the link L of $0 \in \mathcal{A}$. Recall that the fan Σ of Y has support $\mathcal{A}$. The cones of Σ of dimension p correspond to toric strata $Z \subset Y$ of codimension p. These correspond to strata $Z \cap \overline{X} \subset \overline{X}$ of codimension p, which are connected (by our assumption) unless $p = \dim \overline{X}$. We can now construct K from L as follows. Give L the structure of a polyhedral complex induced by the fan Σ. For each top dimensional cell, let $Z \subset Y$ be the corresponding toric stratum, and $k = |Z \cap \overline{X}|$. We replace the cell by k copies, identified along their boundaries. Let $\hat{L}$ denote the resulting CW complex. Note immediately that $\hat{L}$ is homotopy equivalent to the one point union of L and a collection of top dimensional spheres. So $\hat{L}$ has only top reduced rational homology iff L does. Finally let $\hat{L}'$ denote the subdivision of $\hat{L}$ induced by the refinement Σ' of Σ. Then $\hat{L}'$ is the dual complex K of B'. This completes the proof. □

We note the following corollary of the proof.

Corollary 3.3. *In the situation of Thm. 2.5, if in addition $\overline{X} \cap O$ is connected for every orbit $O \subset Y$, then we have an identification*

$$\tilde{H}_{n-1}(L, \mathbb{C}) = \operatorname{Gr}^W_{2n} H^n(X, \mathbb{C}).$$

4 Examples

We say a variety X is *very affine* if it admits a closed embedding in an algebraic torus. If X is very affine, the *intrinsic torus* of X is the torus T with character lattice $M = H^0(\mathcal{O}_X^\times)/k^\times$. Choosing a splitting of the exact sequence

$$0 \to k^\times \to H^0(\mathcal{O}_X^\times) \to M \to 0$$

defines an embedding $X \subset T$, and any two such are related by a translation.

Example 4.1. Let X be the complement of an arrangement of m hyperplanes in $\mathbb{P}^n$ whose stabiliser in $\mathrm{PGL}(n)$ is finite. Then X is very affine with intrinsic torus $T = (\mathbb{C}^\times)^m/\mathbb{C}^\times$, and the embedding $X \subset T$ is the restriction of the linear embedding $\mathbb{P}^n \subset \mathbb{P}^{m-1}$ given by the equations of the hyperplanes. The embedding $X \subset T$ is schön, and a tropical compactification $\overline{X} \subset Y$

is given by Kapranov's visible contour construction, see [HKT1, Sec. 2]. In [AK] it was shown that the link L of $0 \in \mathcal{A}$ has only top reduced homology, and the rank of $H_{n-1}(L,\mathbb{Z})$ was computed using the Möbius function of the lattice of flats of the matroid associated to the arrangement. Thm. 2.5 gives a different proof that the link has only top reduced rational homology. Moreover, in this case $\overline{X} \cap O$ is connected for every orbit $O \subset Y$, and the mixed Hodge structure on $H^i(X,\mathbb{C})$ is pure of weight $2i$ for each i. So we have an identification

$$\tilde{H}_{n-1}(L,\mathbb{C}) = \operatorname{Gr}^W_{2n} H^n(X,\mathbb{C}) = H^n(X,\mathbb{C})$$

by Cor. 3.3.

Example 4.2. Let $X = M_{0,n}$, the moduli space of n distinct points on $\mathbb{P}^1$. The variety X can be realised as the complement of a hyperplane arrangement in $\mathbb{P}^{n-3}$, in particular it is very affine and the embedding $X \subset T$ in its intrinsic torus is schön by Ex. 4.1.

More generally, consider the moduli space $X = X(r,n)$ of n hyperplanes in linear general position in $\mathbb{P}^{r-1}$. The Gel'fand–MacPherson correspondence identifies $X(r,n)$ with the quotient $G^0(r,n)/H$, where $G^0(r,n) \subset G(r,n)$ is the open subset of the Grassmannian where all Plücker coordinates are nonzero and $H = (\mathbb{C}^\times)^n/\mathbb{C}^\times$ is the maximal torus which acts freely on $G^0(r,n)$. See [GeM, 2.2.2]. Thus the tropical variety $\mathcal{A}$ of $X(r,n)$ is identified (up to a linear space factor) with the tropical Grassmannian $\mathcal{G}(r,n)$ studied in [SS]. In particular, for $r = 2$, the tropical variety of $M_{0,n}$ corresponds to $\mathcal{G}(2,n)$, the so called space of phylogenetic trees. For $(r,n) = (3,6)$, the link L of $0 \in \mathcal{A}$ has only top reduced homology, and the top homology is free of rank 126 [SS, 5.4]. Jointly with Keel and Tevelev, we showed that the embedding $X \subset T$ of $X(3,6)$ in its intrinsic torus is schön (using work of Lafforgue [L]) and described a tropical compactification $\overline{X} \subset Y$ explicitly. So Thm. 2.5 gives an alternative proof that L has only top reduced rational homology. Moreover, $\overline{X} \cap O$ is connected for each orbit $O \subset Y$, and the mixed Hodge structure on $H^i(X(3,6),\mathbb{C})$ is pure of weight $2i$ for each i by [HM, 10.22]. So by Cor. 3.3 we have an identification

$$H_{d-1}(L,\mathbb{C}) = \operatorname{Gr}^W_{2d} H^d(X(3,6),\mathbb{C}) = H^d(X(3,6),\mathbb{C})$$

where $d = \dim X(3,6) = 4$. This agrees with the computation of $H^{\cdot}(X,\mathbb{C})$ in [HM].

We note that it is conjectured [KT, 1.14] that $X(3,7)$ and $X(3,8)$ are schön, but in general the compactifications of $X(r,n)$ we obtain by toric methods will be highly singular by [L, 1.8]. The cases $X(3,n)$ for $n \leq 8$ are closely related to moduli spaces of del Pezzo surfaces, see Ex. 4.3 below

Example 4.3. [HKT2] Let $X = X(n)$ denote the moduli space of smooth marked del Pezzo surfaces of degree $9-n$ for $4 \leq n \leq 8$. Recall that a del Pezzo surface S of degree $9-n$ is isomorphic to the blowup of n points in $\mathbb{P}^2$ which are in general position (i.e. no 2 points coincide, no 3 are collinear, no 6 lie on a conic, etc). A *marking* of S is an identification of the lattice $H^2(S,\mathbb{Z})$ with the standard lattice $\mathbb{Z}^{1,n}$ of signature $(1,n)$ such that $K_S \mapsto -3e_0 + e_1 + \cdots + e_n$. It corresponds to a realisation of S as a blowup of n ordered points in $\mathbb{P}^2$. Hence $X(n)$ is an open subvariety of $X(3,n)$ (because $X(3,n)$ is the moduli space of n points in $\mathbb{P}^2$ in *linear* general position). The lattice $K_S^{\perp} \subset H^2(X,\mathbb{Z})$ is isomorphic to the lattice E_n (with negative definite intersection product). So the Weyl group $W = W(E_n)$ acts on $X(n)$ by changing the marking. The action of the Weyl group W on X induces an action on the lattice N of 1-parameter subgroups of T which preserves the tropical variety $\mathcal{A}$ of X in $N_{\mathbb{R}}$. The link L of $0 \in \mathcal{A}$ is described in [HKT2, §7] in terms of sub root systems of E_n for $n \leq 7$.

In [HKT2] we showed that for $n \leq 7$ the embedding $X \subset T$ of X in its intrinsic torus is schön and described a tropical compactification $\overline{X} \subset Y$ explicitly. The intersection $\overline{X} \cap O$ is connected for each orbit $O \subset Y$. So L has only top reduced rational homology by Thm. 2.5, and $H_{d-1}(L,\mathbb{C}) = \operatorname{Gr}^W_{2d} H^d(X(n),\mathbb{C})$ where $d = \dim X(n) = 2n-8$ by Cor. 3.3.

Example 4.4. [MY] Let $\tilde{X} \subset (\mathbb{C}^\times)^{mn}$ be the space of matrices of size $m \times n$ and rank ≤ 2 with nonzero entries. (Thus $\tilde{X}$ is the zero locus of the 3×3 minors of the matrix.) Let $X \subset T$ be the quotient of $\tilde{X} \subset (\mathbb{C}^\times)^{mn}$ by the torus $(\mathbb{C}^\times)^m \times (\mathbb{C}^\times)^n$ acting by scaling rows and columns. In [MY] it was shown that the link L of the origin in the tropical variety $\mathcal{A}$ of $X \subset T$ is homotopy equivalent to a bouquet of top dimensional spheres. Here we give an algebro-geometric interpretation of this result.

A point of X corresponds to n collinear points $\{p_i\}$ in the big torus in $\mathbb{P}^{m-1}$, modulo simultaneous translation by the torus. Let $f\colon X' \to X$ denote the space of lines through the points $\{p_i\}$. The morphism f is a resolution of X with exceptional locus $\Gamma \simeq \mathbb{P}^{m-2}$ over the singular point $P \in X$ where the p_i all coincide. Given a point $(C \subset \mathbb{P}^{m-1}, \{p_i\})$ of X', let q_j be the intersection of C with the jth coordinate hyperplane. We obtain a pointed smooth rational curve $(C, \{p_i\}, \{q_j\})$ such that $p_i \neq q_j$ for all i and j, and the q_j do not all coincide. Conversely, given such a pointed curve $(C, \{p_i\}, \{q_j\})$, let F_j be a linear form on $C \simeq \mathbb{P}^1$ defining q_j. Then we obtain a linear embedding

$$F = (F_1 : \cdots : F_m)\colon C \subset \mathbb{P}^{m-1}$$

which is uniquely determined up to translation by the torus.

We construct a compactification $X \subset \overline{X}$ using a moduli space of pointed curves. Let $\overline{X}'$ denote the (fine) moduli space of pointed curves $(C, \{p_i\}_1^n, \{q_j\}_1^m)$ where C is a proper connected nodal curve of arithmetic genus 0 (a union of smooth rational curves such that the dual graph is a tree) and the p_i and q_j are smooth points of C such that

(1) $p_i \neq q_j$ for all i and j.

(2) Each end component of C contains at least one p_i and one q_j, and each interior component of C contains either a marked point or at least 3 nodes.

(3) The q_j do not all coincide.

(The moduli space $\overline{X}'$ can be obtained from $\overline{M}_{0,n+m}$ as follows: for each boundary divisor $\Delta_{I_1,I_2} = \overline{M}_{0,I_1 \cup \{*\}} \times \overline{M}_{0,I_2 \cup \{*\}}$ we contract the ith factor to a point if $I_i \subseteq [1,n]$ or $I_i \subsetneq [n+1, n+m]$.) Define the *boundary* B of $\overline{X}'$ to be the locus where the curve C is reducible. It follows by deformation theory that $\overline{X}'$ is smooth with normal crossing boundary B. The construction of the previous paragraph defines an identification $X' = \overline{X}' \setminus B$. The desired compactification $X \subset \overline{X}$ is obtained from $X' \subset \overline{X}'$ by contracting $\Gamma \subset X'$.

Assume without loss of generality that $m \leq n$. Consider the resolution $f \colon X' \to X$ of X with exceptional locus $\Gamma \simeq \mathbb{P}^{m-2}$ described above. By [GoM, Thm. II.1.1*] since $2 \dim \Gamma \leq \dim X$ and X is affine it follows that X' has the homotopy type of a CW complex of dimension $\dim X$. Hence by Thm. 3.1 the dual complex K of the boundary B has only top rational homology, and $\tilde{H}_{d-1}(K, \mathbb{C}) = \operatorname{Gr}_{2d} H^d(X', \mathbb{C})$ where $d = \dim X' = m+n-3$.

The compactification $\overline{X}$ of X is a tropical compactification $\overline{X} \subset Y$ of $X \subset T$ such that $\overline{X} \cap O$ is connected for each orbit $O \subset Y$. This is proved using the general result [HKT2, 2.10]. The toric variety Y corresponds to the fan Σ with support $\mathcal{A}$ given by [MY, 2.11]. In particular, it follows that K is a triangulation of the link L. Hence we obtain an alternative proof that L has only top reduced rational homology, and a geometric interpretation of the top homology group.

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Paul Hacking, Department of Mathematics, University of Washington, Box 354350, Seattle, WA 98195; `hacking@math.washington.edu`