Let $X$ be a smooth projective variety. To $X$ we associate cycle classes $c_1(X), c_2(X), \ldots, c_n(X)$, where $n = \dim X$. The $c_i(X)$ are defined via the tangent bundle $T_X$, where $c_i(X)$ is the class of a bunch of codimension $i$ subvarieties, e.g. $c_1(X)$ is a divisor class and $c_n(X)$ is the class of a sum of points. We have $c_1(X) = -K_X$, the negative of the canonical class of $X$, and over $\mathbb{C}$, $c_n(X) = e$ where $e$ is the topological Euler characteristic, $e = 1 - b_1 + b_2 - \cdots + b_{2n}$ where the $b_i$ are the Betti numbers.

Out of these classes we get numbers. For example, assuming $\dim X = n$, $c_n(X) = (-1)^n K_X^n$ and we can consider $c_2(X)c_{n-2}(X)$, etc. Try all combinations and we get the Chern numbers of $X$. Under deformations these Chern numbers will be preserved.

Now we can talk about moduli $= M_{\text{Chern#}}$. The main geography question is:

When is $M_{\text{Chern#}} \neq \emptyset$?

**Example** Let $\dim X = 1$ (curves). Then $c_1(X) = -K_X = 2 - 2g$, where $g$ is the genus (number of handles). For $g \geq 0$ let $M_g$ denote the moduli space (parameter space) of curves of genus $g$. We can see that $M_g \neq \emptyset$ by considering hyperelliptic curves $y^2 = \prod_{i=1}^{2g+2}(x - a_i)$ where $a_i \neq a_j$.

What about surfaces ($\dim X = 2$) where $X$ is minimal? We will use the classification of Enriques. Let $\kappa$ be the Kodaira dimension of $X$. Facts we will use:

1. If $\bar{X}$ is the blowup of $X$ at $p$ then
   \[ c_1^2(\bar{X}) = c_1^2(X) - 1 \]
   \[ c_2(\bar{X}) = c_2(X) + 1 \]

2. If $X$ is a surface, $B$ is a curve, and $\pi : X \to B$ is a fibration with general fiber $F$ a smooth projective curve, then
   \[ e(X) = e(B)e(F) + \sum_s (e(F_s) - e(F)) \]
   where the sum is over $s \in B$ such that the fiber $F_s := \pi^{-1}(s)$ is singular.

3. If $X$ is a smooth projective surface and $G$ is a finite group acting freely on $X$, then
   \[ |G|c_1^2(X/G) = c_1^2(X) \]
   \[ |G|c_2(X/G) = c_2(X) \]

Also, $p_g = h^0(K_X)$ and $q = h^1(\mathcal{O}_X)$.

We consider cases:

($\kappa = -\infty$) In this case $X$ is birational to $C \times \mathbb{P}^1$ where $C$ is any smooth projective curve. The minimal such $X$ are $\mathbb{P}_C(E)$, where $E$ is a rank 2 vector bundle on $C$. Then
\[ c_1^2(X) = 8(1 - g) \]
\[ c_2(X) = 4(1 - g) \]

($\kappa = 0$) Here there are four cases:

(i) $X$ is a $K3$ surface. Then $c_1^2 = 0$ and $e(X) = 24$. 


(ii) $X$ is an Enriques surface (in which case we have a 2 : 1 map from a $K3$). Here $c_1^2 = 0$ and $e(X) = 12$.

(iii) $X$ is an Abelian surface. Then $c_1^2 = 0$ and $e(X) = 0$.

(iv) $X$ is a bi-elliptic surface, then $c_1^2 = 0$ and $e(X) = 0$.

($\kappa = 1$) Now we have a fibration $\pi : X \to B$ to a smooth projective curve $B$ where a general fiber is a curve of genus 1 (i.e. $\pi$ is an elliptic fibration). Here $K_X$ is a bunch of fibers and so $K_X^2 = 0 = c_1^2$ since the self intersection of fibers is 0. In this case

$$e(X) = \sum_s e(F_s) \geq 0.$$  

For $e(X) = 0$ we can take $E \times C$ for any curve $C$.

If $e > 0$ there are two cases: Recall Noether’s formula:

$$12\chi(O_X) = c_1^2 + c_2.$$  

The cases are

(i) $\chi \geq 3$ (simple). We can construct $X \to B$ as a pullback of a rational elliptic fibration.  
(Note: If we try the same approach for $\chi < 3$ the result has $\kappa < 1$.)

(ii) $\chi = 1, 2$ (this case is hard). There are Dolgachev surfaces: $X_{2,q} \to \mathbb{P}^1$ an elliptic fibration. It is simply connected and has two multiple fibers of order 2, $q$. Here $\chi = 1$, $p_g = 0$.

($\kappa = 2$)

Now we are in the case where $X$ is a surface of general type (and we will assume $X$ is minimal over $\mathbb{C}$). The main question we wish to answer is: When is $M_{K_X^2,e} \neq \emptyset$? We have the following restrictions on $K_X^2, e$ for the existence of $X$.

(1)

$$K_X^2 > 0$$  

(2)

$$e > 0$$  

(3)

$$K_X^2 + e \equiv 0(12)$$  

(4)

$$5K_X^2 - e(X) + 36 \geq 12q \geq 0$$  

(From Noether’s inequality $2p_g - 4 \leq K_X^2$.)

**Theorem** (Bogomolov-Miyaoka-Yau (1977) Inequality).

(5)

$$K_X^2 \leq 3e(X)$$  

These are believed to be the only restrictions.

Comments:

(i) If $\text{Char}(\mathbb{K}) > 0$ then (5) and (2) are not necessarily true.

(ii) One can prove that for any complete intersection in $\mathbb{P}^{r+2}$, $c_1^2 < 2c_2$.

Notice that $c_1^2/c_2 \in [-1/5, 3]$. By the 1950’s there were examples only with $c_1^2/c_2 < 2$. Then Hirzebruch proved the following. Write

$$\mathbb{H} = \{z \in \mathbb{C} : |z| < 1\}$$  

$$\mathbb{B} = \{(z, w) \in \mathbb{C}^2 : |z|^2 + |w|^2 < 1\}$$
Suppose a discrete group $\Gamma$ acts freely and discontinuously on $\mathbb{H} \times \mathbb{H}$. Let $X = (\mathbb{H} \times \mathbb{H})/\Gamma$ denote the quotient (a complex manifold) and assume $X$ is compact. Then $X$ satisfies $c_1^2 = 2c_2$. Similarly for $X = \mathbb{B}/\Gamma$ we have $c_1^2 = 3c_2$ (examples of this type were constructed by Hirzebruch and Borel).

**Theorem.** (Hirzebruch’s signature formula) The signature $\sigma$ of $X$ equals $1/3(c_1^2 - 2c_2)$.

People try to populate the range $2 < c_1^2/c_2 < 3$ (where $\sigma > 0$).

When Yau proved the BMY inequality he also proved that $X$ satisfies $c_1^2 = 3c_2$ if and only if $X = \mathbb{B}/\Gamma$. In particular $\pi_1(X) = \Gamma$ is not trivial.

**Sketch of the proof of the inequalities:**

**Theorem.** If $X$ a minimal surface of general type then $2p_g - 4 \leq K_X^2$.

We can assume $p_g \geq 3$. Let $F : X \longrightarrow \mathbb{P}^{p_g-1}$ be the rational map defined by the linear system $|K_X|$. There are two cases: either $|K_X|$ is composed to a pencil (that is, the image of $F$ is a curve) or the image of $F$ is a surface. We will assume $|K_X|$ contains a smooth projective curve $C$. Then we are in the second case. We have an exact sequence

$$0 \rightarrow \mathcal{O}_X(K_X - C) \rightarrow \mathcal{O}_X(K_X) \rightarrow \mathcal{O}_C(K_X|_C) \rightarrow 0.$$  

The associated left exact sequence of global sections gives $p_g - 1 \leq h^0(C, K_X|_C)$. By the adjunction formula, $K_X|_C = K_C - C|_C$ so $C|_C$ is a special divisor on $C$. We have Clifford’s inequality

$$h^0(D) \leq (\deg D)/2 + 1,$$

so we obtain $p_g - 1 \leq K_X^2/2 + 1$ as required.

**Proof of $B - M - Y$ inequality.**

This is hard. First an example.

**Example:** Let $\{L_1, \ldots, L_d\}$ be $d$ lines in $\mathbb{P}_C^2$. Say $\tau_k = \#\{k - \text{uple points}\}$. Then BMY implies that over $\mathbb{C}$, if $\tau_d = \tau_{d-1} = 0$ then

$$2\tau_2 + \tau_3 \geq 3 + d + \sum_{k \geq 4} (k - 4)\tau_k.$$

In particular double or triple points exist.

$\mathbb{P}^2_\mathbb{R}$ is really a surface with $e(\mathbb{P}^2_\mathbb{R}) = 1$. It can be proven that

$$0 \leq \left( \sum_{k \geq 3} (k - 3) \right) p_k = -3 - \sum_{k \geq 2} (k - 3)\tau_k,$$

where $p_k$ is the number of two cells bounded by $k$ one-cells. This number is 0 if and only if we have only triangles (simplicial).

Now back to the proof. This is due to the magic of Miyaoka

Suppose (for a contradiction) that $\alpha := c_2/c_1^2 < 1/3$ and write $\beta = 1/4(1 - 3\alpha) > 0$. Set key $= S^u \Omega_X \otimes \mathcal{O}_X(-n(\alpha + \beta)K_X)$. Then the Riemann-Roch formula gives an inequality

$$h^0(\text{key}) + h^2(\text{key}) \geq \frac{1}{6 \cdot 16} (3\alpha^2 - 22\alpha + 7)K_X^2 n^3 + O(n^2).$$

Hence for $n \gg 0$ we have $h^0(\text{key}) + h^2(\text{key}) > 0$ (note $3\alpha^2 - 22\alpha + 7 > 0$ since $\alpha < 1/3$). This can be used to obtain a contradiction.

The proof of Yau (assuming $K_X$ is ample) shows that there exists a Kähler-Einstein metric on $X$. Then

$$c_1^2 - 3c_2 = \int_X f d\text{vol}$$
where \( f \geq 0 \), so \( c_1^2 - 3c_2 \geq 0 \).

Now we draw a map of the geography for minimal surfaces of general type.